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On Certain Initial Techniques to Model Singular Functions

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Abstract

Nowadays many researches are devoted to pathological mathematical objects of real analysis such as the Moran and Cantor-like fractal sets, functions with complicated local structure, as well as their generalizations and various applications. For example, these applications include the development of fractal multiformalism, general fractal measures and dimensions, as well as Hewitt-Stromberg measures and homogeneous Moran measures, physical and economical modeling, etc. According to historical way of investigations in mathematics, it is important the question on the methodological tools used by classical mathematicians to advance the study of pathological functions. The present survey is devoted to examples and the main techniques for modeling mainly singular functions introduced before 2000 in papers indexed in Scopus. Since the later research in this topic must more explanations, the rest examples will be considered in next papers of the author. The main considered techniques to construct singular functions are following: using systems of functional equations; applying Markov chains and distribution function; using various expansions of arguments and values of a function; certain geometrical iteration procedures; using auxiliary relations for the geometric construction of the graph of a function; applications of auxiliary maps, compositions of functions, and iterated function systems.

Keywords

Singular functions, Systems of functional equations, Compositions of functions, Expansions of real numbers, Probability distributions

AMS Classification: 11K50, 11K55, 26A30

Article History

Received: 09 July 2025 Revised: 19 September 2025

Accepted: 15 October 2025 Available Online: 28 November 2025

1. Introduction

Functions whose derivative is equal to zero almost everywhere on the domain in the sense of the Lebesgue measure, are singular functions. In real analysis, we often deal with probability distributions on the unit interval. According to this way, such singular functions have zero derivative on subsets of $(0,1)$ with the full Lebesgue measure. In addition, the key characteristic of singular functions is their complicated local structure. This characteristic can be explained by the existence of at least one point in an arbitrarily small neighborhood of the domain, where the derivative is infinite. The second peculiarity is ([1]) the fact that such functions form a set of the second Baire category in the metric space of all continuous monotone functions with supremum metric.

Singular functions belong to a class of mathematical pathologies in real analysis. The mentioned class contains non-differentiable functions (whose derivative does not exist or is infinite), nowhere monotonic functions, and fractal sets. These objects can be characterized by a wide spectrum of their applications and by some connections. For example, in [2], the following was noted:

"Singular functions have been attracting more attention in connection with studies of chaos and fractal. … by means of simple physical models, … physical meanings to the classical singular functions, de Wijs' fractal, Lebesgue's singular function and Takagi's function. It is shown that the integral of a quantity which forms de Wijs' fractal can be expressed by Lebesgue's singular function. Takagi's function is found to be proportional to the flux which is induced when a uniform distribution changes to de Wijs' fractal." [2].

Really [2], using a constant voltage applied to a one-dimensional electric resistor of the unit length, a voltage $V(x)$ at the point x can be calculated by the equality

$$V(x) = V \cdot C(x) \tag{1}$$

where $C(x)$ is the Cantor singular function. Moreover, in the theory of turbulence, one can prove that "the y-component of the velocity field is proportional to" the last-mentioned singular function, as well as "in the case that the vorticity changes with time, the flux of the vorticity will be proportional "to the continuous non-differentiable Takagi function [2] under certain conditions.

As for other applications, let us remark that in the theory of trigonometric series, the Riesz products can be singular functions under some conditions. The Minkowski question-mark function is a bijection between rational numbers and numbers with finite dyadic representations. In fractal theory, singular functions can be conjugating homeomorphisms or Perron-Frobenius measures according to the theory of wavelets as certain special functions, that provide a satisfactory answer to the scale problem under decomposable events in nature such as self-similar behavior in iterated processes or fingerprints [3].

Classics of mathematicians began to model singular, nowhere monotonic, and non-differentiable functions. For example, the first example of a continuous nowhere differentiable function was constructed by du Bois-Reymond and was published in 1875. The first example of singular functions was independently published by Scheefer and Cantor later (twelve years after the Weierstrass non-differentiable function) [3].

Modeling examples of pathological real functions with complicated local structure was continued and extended by a number of researchers including classics of mathematics and their closest followers. For example, such scientists as Weierstrass, Dini, and Takagi, as well as Cèsaro, Hellinger, and Viader, Lebesgue, etc., have investigated the noted objects.

Nowadays many investigations are devoted to the Moran or Cantor-like fractal sets and their applications (for example, see [4-6] and references therein), to generalizations of functions with complicated local structure ([7-8], etc.) and their connections with fractal analysis (in fractal multiformalism (for the main explanations and the motivation, see [9] and references therein; also, see [10-13]), in general fractal measures and dimensions [14-17], and in Hewitt-Stromberg measures and homogeneous Moran measures [18-21], etc.) and with various mathematical areas (for example, see [22] for duality functions for pairs of aggregation operators on the unit interval, see [2] for physical meanings to the classical singular functions, etc. [23,24]).

In this survey, we consider examples and the main techniques for modeling functions with complicated local structure (mainly singular functions) published before 2000 from papers in Scopus. The main goal is to show what methodological instruments were used by classics of mathematics for the progress in the area of pathological functions. In the next paper, the attention will be given investigations of singular functions such that were published after 2000.

The motivation of this study can be explained by extensions of the theory of functions with complicated local structure in modern research (for example, see [25-28], etc.), by connections such mathematical objects with various expansions of real numbers, as well as by their importance for educational goals according to ideas of Henri Poincarè [28].

2. The First Examples of Singular Functions

Let us begin with earlier examples of singular functions.

One can note that the history of continuous monotone singular functions begins in 1883-1904 with corresponding examples modeled by Cantor in [29] in 1884 (this example was (see [30] and references below) also considered by Lebesgue in [31] in 1904, by Scheeffer in 1884, and by Vitali in 1905), as well as by Minkowski in [32] in 1904.

Let x be a number represented in terms of the ternary expansion of real numbers, i.e.,

$$x=\sum_{k=1}^{\infty}\frac{i_k(x)}{3^k} \qquad (2)$$

where $i_k \in \{0,1,2\}$.

The classical Cantor function is a continuous (but not absolutely continuous) and increasing function, which maps the classical Cantor set $C_0=\left\{x: x=\sum_{k=1}^{\infty}\frac{u_k}{3^k},\ u_k\in\{0,2\}\right\}$ onto $[0,1]$and is constant on $[0,1]/C_0$.

Let us describe a definition of this function by systems of functional equations. Really [33], for the Banach space of all uniformly bounded real-valued functions on $[0,1]$ with the supremum norm, this function is a unique solution of a system of functional equations of the form

$$\begin{cases} f(x)=\frac{1}{2}f(3x) & \textit{whenever } x\in\left[0,\frac{1}{3}\right] \\ f(x)=\frac{1}{2} & \textit{whenever } x\in\left(\frac{1}{3},\frac{2}{3}\right) \\ f(x)=\frac{1}{2}+\frac{1}{2}f(3x-2) & \textit{whenever } x\in\left[\frac{2}{3},1\right] \end{cases} \qquad (3)$$

Finally, one can note an analytic representation by the form, which in our opinion, is one of the best description according to [33]):

$$C(x)=\frac{1}{2^{n(x)}}+\frac{1}{2}\sum_{k=1}^{n(x)-1}\frac{i_k(x)}{2^k} \qquad (4)$$

where

$$n(x)=\begin{cases} 1 \ \textit{whenever} & n=\min_k\{k: i_k(x)=1\} \\ \infty \ \textit{whenever} & \text{there is no such positive integer } n \end{cases} \qquad (5)$$

Now the Cantor function is [30] also called Lebesgue's singular function, the Cantor–Vitali function, or the Devil's staircase, as well as the Cantor staircase function or the Cantor–Lebesgue function. For more detail survey, see [33].

Finally, one can note on the geometric approach to construct the graph of C. An algorithm can be noted as following:

The first step. Let us construct a partition of the abscissa axis into three intervals $\left[0,\frac{1}{3}\right],\left(\frac{1}{3},\frac{2}{3}\right),\left[\frac{2}{3},1\right]$ and a partition by $\left[0,\frac{1}{2}\right],\left[\frac{1}{2},1\right]$ for the ordinate axis.

Put

$$C\left(\left(\frac{1}{3},\frac{2}{3}\right)\right)=\left\{\frac{1}{2}\right\},$$

i.e., $C(x)=\frac{1}{2}$ for all $x\in\left(\frac{1}{3},\frac{2}{3}\right)$.

In the second step, we have a partition by

$$\left[0,\frac{1}{9}\right],\left(\frac{1}{9},\frac{2}{9}\right),\left[\frac{2}{9},\frac{3}{9}\right]$$

and

$$\left[\frac{6}{9},\frac{7}{9}\right],\left(\frac{7}{9},\frac{8}{9}\right),\left[\frac{8}{9},1\right]$$

for the closed intervals (segments obtained in the first step) for the abscissa axis, as well as by

$\left[0,\frac{1}{4}\right],\left[\frac{1}{4},\frac{2}{4}\right],\left[\frac{2}{4},\frac{3}{4}\right],\left[\frac{3}{4},1\right]$

for the ordinate axis.

Hence,

put

$$C\left(\left(\frac{1}{9},\frac{2}{9}\right)\right)=\left\{\frac{1}{4}\right\}$$

and

$$C\left(\left(\frac{7}{9},\frac{8}{9}\right)\right)=\left\{\frac{3}{4}\right\}.$$

Applying the algorithm, we get the well-known graph of the function C (for example, see [30,33]):

So, for the Cantor function, one can remark that the used for its modeling techniques are following: using different expansions of real numbers for arguments and values (in our case, the ternary and binary representations are such expansions), and later there are used systems of functional equations.

Let us consider the classical singular Minkowski question mark function. As noted above, this function was introduced in [32].

The first definition of the Minkowski question mark functions is related with the Stern–Brocot tree and the Farey sequence (the notions of number theory). An algorithm is following:

Put $h(0)=0$ and $h(1)=1$.

For future modeling of values, let us apply the following formula:

$$h\left(\frac{a+b}{c+d}\right)=\frac{1}{2}\left(h\left(\frac{a}{b}\right)+h\left(\frac{c}{d}\right)\right) \tag{6}$$

where $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers such that the condition $ad-bc=1$ holds.

Whence, we have:

If $h\left(\frac{0}{1}\right)=0$ *and* $h\left(\frac{1}{1}\right)=1$, *then*

$$h\left(\frac{0+1}{1+1}\right)=h\left(\frac{1}{2}\right)=\frac{1}{2}\left(h\left(\frac{0}{1}\right)+h\left(\frac{1}{1}\right)\right)=\frac{1}{2}.$$

Use obtained values for calculations of related values, i.e.,

$$h\left(\frac{0+1}{1+2}\right)=h\left(\frac{1}{3}\right)=\frac{1}{2}\left(h\left(\frac{0}{1}\right)+h\left(\frac{1}{2}\right)\right)=\frac{1}{4},$$

$$h\left(\frac{1+1}{1+2}\right)=h\left(\frac{2}{3}\right)=\frac{1}{2}\left(h\left(\frac{1}{1}\right)+h\left(\frac{1}{2}\right)\right)=\frac{3}{4}.$$

In the third step, we get $h\left(\frac{1}{4}\right)=\frac{1}{8}$

and $h\left(\frac{3}{4}\right)=\frac{7}{8}$.

By analogy, one can calculate values for $\frac{1}{5}$, $\frac{2}{5}$, and $\frac{3}{5}$, as well as $\frac{4}{5}$.

In the nth step, we obtain values for

$$0=\frac{0}{n},\frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n},\frac{n}{n}=1.$$

The noted iterative procedure defines the Minkowski function at all rational points. It is easy to see that this function maps the set of all rational numbers into the set of numbers with finite binary expansions.

Using the notion of the density of sets and the monotonicity, one can prove that the Minkowski function is continuous and $h:[0,1]\to[0,1]$.

Let us note on the second quite often used definition. This definition is based on different numeral systems and is analytical, as well as was introduced by Denjoy in [34] in 1938.

Let an argument x be defined in terms of continued fraction expansions, i.e.,

$$x=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cdots}}}\coloneqq[0;a_1,a_2,a_3,\cdots] \tag{7}$$

Here a_k, where $k=1,2,3,\ldots$, are positive integers. Then values of the Minkowski question mark function can be represented in terms of alternating binary expansions, i.e.,

$$h(x)=\sum_{k=1}^{\infty}(-1)^{k-1}2^{1-(a_1+a_2+\ldots+a_k)} \tag{8}$$

The graph of the Minkowski function is well-known and was noted in many researches or surveys (for example, see [35,36]).

So, the first examples of singular functions were defined in terms of various expansions of real numbers, geometrical relationships of the graph, or functional equations. But the first techniques contain methods of number theory and geometry and have an abstract character.

In the next section, the main attention will be given to later definitions of other singular functions.

3. Known Examples of Singular Functions (Introduced in the Period until 2000)

In this section, the main attention will be given to investigations of functions of one variable such that were published in papers indexed in Scopus.

This considering period can be characterized by few published fundamental papers marked the beginning of a whole cycle of research that continues today.

In [37], the main attention was given to an elementary construction of a continuous and strictly monotone function $f:[0,1]\to[0,1]$ such that for a suitable set S, both the complement of S and the image of S under f are sets of zero measure. It is used iterated approach. That is, it is defined a function of the form

$$f\left(x_k^n\right)=\frac{k}{2^{n-1}} \tag{9}$$

where $k=0,1,2,\ldots,2^{n-1}$ for positive integers n, as well as for positive integers n and j, the arguments defined by the following:

$x_0^1\coloneqq 0$ *and* $x_1^1=1$;

$0=x_0^n<x_1^n<x_2^n<\ldots<x_k^n<\ldots<x_{2^{n-1}}^n$,

$$x_{2j-1}^{n+1}=\frac{nx_{j-1}^n+x_j^n}{n+1}, \text{ and } x_{2j}^{n+1}=x_j^n \tag{10}$$

In [38], Salem investigated values for the Fourier-Stieltjes coefficients of continuous singular monotonic functions which are of the Cantor type (i.e., constant functions in each interval contiguous to a perfect set of measure zero). Giving a brief survey on related researches, one can note a brief mention about examples of singular monotonic functions given by Wiener, Wintner, and Schaeffer. In this paper, Salem have constructed the main example of singular function by Wiener's and Wintner's technique.

Let $\mu<1$ be a fixed number, $(d+1)$ and $0=\lambda(0),\lambda(1)\le\mu,\ldots,\lambda(d+1)\le\mu$ be positive integers such that the condition $\lambda(1)+\lambda(2)+\lambda(3)+\ldots+\lambda(d+1)=1$ hold.

Using auxiliary parameters, we get the final form

$$\Phi(x)=\sum_{i=0}^{\theta_1}(\lambda(i))+\sum_{k=2}^{\infty}\left(\left(\sum_{i=0}^{\theta_k}(\lambda(i))\right)\prod_{j=1}^{k-1}\lambda(\theta_j+1)\right) \tag{11}$$

which using the equalities $\lambda(1)=\lambda(2)=\lambda(3)=...=\lambda(d+1)=\frac{1}{d+1}$ can be simplified to the form

$$\Phi(x)=\sum_{k=1}^{\infty}\frac{\theta_k}{(d+1)^k}.$$

In 1940th, the main useful classical example of singular function was introduced by Salem in [39]. In the last paper, the Salem function was constructed, as well as for Fourier-Stieltjes coefficients and the singularity of the Minkowski function, proofs are given.

Let us define the argument of the Salem function.

Let $q>1$ be a fixed positive integer number and $i_k\in\{0,1,2,...,q-1\}$ for all positive integers k; then for a real number $x\in[0,1]$, an expansion of the form

$$x=\frac{i_1}{q}+\frac{i_2}{q^2}+\frac{i_3}{q^3}+...+\frac{i_k}{q^k}+..., \quad (12)$$

is called *a q-expansion of* $x\in[0,1]$.

Let $x\in[0,1]$ be an argument and $P=(p_0,p_1,...,p_{q-1})$ be a fixed probability vector with positive coordinates , and

$$1>\beta_{i_k}=\begin{cases}0 & \textit{whenever } i_k=0\\ p_0+p_1+...+p_{i_k-1} & \textit{whenever } i_k\neq 0\end{cases}; \quad (13)$$

then an analytical representation of the classical Salem function is of the following form

$$S(x)=\beta_{i_1}+\sum_{k=2}^{\infty}\left(\beta_{i_k}\prod_{r=1}^{k-1}p_{i_r}\right) \quad (14)$$

and can be obtained by the definition of a distribution function (for example, see [40-43] and references therein). But this technique for the definition appeared later. In [39], to construct this function, an geometrical approach was applied. That is [39], in the plane, we have "the straight line PQ joining the point P of cartesian coordinates x, y, to the point Q of cartesian coordinates $x+\Delta x$, $y+\Delta y$" with positive Δx and Δy. Let p_0 and p_1 be two different essentially positive numbers for which the condition $p_0+p_1=1$ holds. Then let us consider the point R having the following coordinates:

$x+\frac{\Delta x}{2}$ and $y+p_0\Delta y$.

That is, "the horizontal distance between P, R or between Q, R is $\frac{\Delta x}{2}$, while the vertical distance between P, R is" $p_0\Delta y$, and between R and Q is $p_1\Delta y$. ". If we replace the straight line PQ by the broken line PRQ, we will say that we perform on PQ the transformation" $T(p_0,p_1)$ [39].

So, the definition iteration is following [39]:

For $x\in[0,1]$, let us define $f(x)=x$, i.e., "to say represented by the straight line OA joining the origin O to the point $A(1,1)$".

If we use on OA the transformation $T(p_0,p_1)$, then we obtain "a broken line consisting of two straight lines and representing an increasing function $f_1(x)$. Let us perform on each of those two straight lines the transformation" $T(p_0,p_1)$.

By analogy, we have a broken line consisting of 4 straight lines and representing an increasing function $f_2(x)$.

For the k th step, we get a strictly increasing function $f_k(x)$ with $f_k(0)=0$ and $f_k(1)=1$ "represented by a polygonal line consisting of 2^k straight lines, the vertices having for abscissa the points" $\frac{j}{2^k}$ for $j=1,2,3,...,2^k-1$.

So, using that $\max\{p_0,p_1\}<1$, we get

$$S(x) = \lim_{k \to \infty} f_k. \quad (15)$$

It is easy to see that in this case $q = 2$ [39].

Now we consider the paper [44], where Kober presented several interesting general statements about relations singular functions of bounded variation and explained the significance of singular functions by the fact that “any function of bounded variation is the sum of an absolutely continuous function and a singular function” [44].

Theorem 1. ([44]). *If $y = f(x)$ is a non-decreasing singular function, not reducing to a constant, then the inverse function $x = f^{-1}(y)$ is singular.*

As noted in [44], singular functions of bounded variation over the closed interval $[0, a]$ are the step-functions or, in a general case, the jump-functions, and “the sum of the moduli of their jumps is equal to their total variation, or converges to it when there is an infinity of discontinuities” [44].

Lemma 1. ([44]). *If $x(t)$ and $y(t)$ are functions of bounded variation over the closed interval $[0, a]$, as well as the equalities $x(t) = e(t) + m(t)$ and $y(t) = n(t) + v(t)$ hold, where $e(t), n(t)$ are absolutely continuous and $m(t), v(t)$ are singular, then $L_0(x, y)$, the length of the arc of the curve*

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad (16)$$

joining the points $(x(0), y(0))$ and $(x(a), y(a))$, is

$$L_0(x, y) = L_0(e, n) + L_0(m, v), \quad (17)$$

where the function $L_0(e, n)$ is absolutely continuous and $L_0(m, v)$ is singular.

In 1950s, the mentioned Salem function was modeled and studied by techniques of probability theory, i.e., using Markov chains and distribution function (see [45,46]). Really (for example, see [46] and references therein), assume that (x_k) is a chain with a finite number of states, $0, 1, 2, ..., U-1$, and

$$X = \sum_{i=1}^{\infty} \frac{x_i}{U^i} \quad (18)$$

is a random variable with an associated distribution function of the form

$$F(x) = \Pr ob\{X < x\} \quad (19)$$

and

$$F(A) = \Pr ob\{X \in A\} = \int_A dF(x) \quad (20)$$

In 1978, Takács ([47]) used an analytical approach and modeled a singular strictly increasing continuous function with the arguments defined in terms of certain binary series and values defined by certain series, i.e., for a fixed positive real number $\rho > 1$ and sequences $(a_k)_{k=0}^{\infty}$ of positive integers such that $a_0 < a_1 < a_2 < ... < a_k < ...$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{\rho^k}{(1+\rho)^{a_k}} \text{ with } f(0) = 0 \quad (21)$$

and

$$f(1) = 1 \text{ for } x = \sum_{k=0}^{\infty} \frac{1}{2^{a_k}}. \quad (22)$$

Noting an analytical approach, let us consider a technique used in [48], where C_e is the classical Cantor function $C(x)$ extended on $(-\infty, +\infty)$, i.e.,

$$C_e(x) = \begin{cases} 0 & whenever\, x \le 0 \\ C(x) & whenever\ x \in (0,1) \\ 1 & whenever\, x \ge 1 \end{cases}. \quad (23)$$

Suppose $M = \{m_1, m_2, m_3, ..., m_k, ...\}$ is an arbitrary countable dense set of real numbers. Then using the notion of the composition of functions, the following example of a continuous strictly increasing singular function can be constructed (see [48]):

$$g(x) = \sum_{k=1}^{\infty} \frac{C_e\left(2^k (x - a_k)\right)}{2^k}. \quad (24)$$

Let us describe a more complicated approach to model functions with zero derivative by compositions of functions and iterated sequences such that was introduced in [49]. Let us begin with auxiliary definitions and let us note some general theorems.

A real function f is called a smooth function at the point $x \in (a,b)$ whenever the following condition holds:

$$f(x+h) + f(x-h) - 2f(x) = o(h), \quad (25)$$

and is called an almost smooth function at $x \in (a,b)$ whenever

$$f(x+h) + f(x-h) - 2f(x) = O(h), \quad 26)$$

where $a < x - h < x + h < b$. In addition, A real function f is called an uniformly almost smooth function at the point $x \in (a,b)$ for $-\infty \le a < b \le +\infty$ whenever there exists a number K such that the condition

$$| f(x+h) + f(x-h) - 2f(x) | < Kh \quad (27)$$

holds for all x and h with $a < x - h < x + h < b$. [49].

Theorem 2. ([49]). *There exist strictly increasing, uniformly almost smooth functions with a derivative, which, wherever it exists, is equal to* 0 *or* ∞.

Theorem 3. ([49]). *There exist non-constant monotonic uniformly almost smooth functions on* $[0,1]$ *whose derivative vanishes on an open set of measure 1.*

Theorem 4. ([49]). *There exists a strictly increasing singular uniformly smooth function.*

Theorem 5. ([49]). *There exists a non-constant monotonic uniformly smooth function on* $[0,1]$ *whose derivative vanishes on an open set of the full measure.*

For modelling functions with complicated local structure, one can note the following algorithm using iterated function systems and compositions of functions.

Let $m_1, m_2, m_3, ..., m_{k-1}$ be a fixed tuple of positive integers.

Then, in the first step, for $x \in [0,1]$, we assume that $f_0(x) = g_0(x) = x$, $g(x) = -x(2x-1)(x-1)$, and

$$f_1(x) = f_0(x) + g_1(x) = x + \frac{1}{m_1} g(m_1 x - u) \quad (28)$$

for $u = \overline{0, m_1 - 1}$.

For the $(k-1)$th and k th steps, we have a function of the form

$$f_{k-1}(x) = g_0(x) + g_1(x) + g_2(x) + ... + g_{k-1}(x) \quad (29)$$

being a continuous strictly monotonic function. In addition,

$$f_k(x) = f_{k-1}(x) + g_k(x) = f_{k-1}(x) + a_{ik} g(m_k x - u) \quad (30)$$

where a_{ik} is the greatest number that does not exceed $\frac{1}{m_k}$ and for this number f_k is an increasing function. Here $g_k(x) = 0$ whenever $x \in \left\{0 = \frac{0}{m_k}, \frac{1}{m_k}, \frac{2}{m_k}, ..., \frac{u}{m_k}, ..., \frac{m_k}{m_k} = 1\right\}$.

It is easy to see [49] that the sequence (f_k) converges uniformly to a strictly increasing function f.

Finally, let us note that the paper [50] continues the development of representations of singular functions by functional equations. It is considered the following systems of functional equations

$$\begin{cases} f\left(\frac{x}{x+1}\right) = af(x) \\ f\left(\frac{1}{2-x}\right) = a + (1-a)f(x) \end{cases} \qquad (31)$$

and

$$\begin{cases} f\left(\frac{x}{x+1}\right) = af(x) \\ f\left(\frac{1}{x+1}\right) = 1 - (1-a)f(x) \end{cases} \qquad (32)$$

where $x \in [0,1]$, $t \in (0,1)$ is a given parameter, and $f:[0,1] \to (-\infty, +\infty)$ is an unknown map.

Systems (31) and (32) have an unique continuous bounded solution under a fixed parameter t in the class of defined and bounded on $[0,1]$ functions. Moreover, if $t = 2$, then the mentioned classical Minkowski function is a unique solution of (32). For proofs there are used Banach's fixed point theorem, properties of the Farey fractions, etc. In addition, in [50], some attention was also given to constructions of such functions by the Farey fractions by analogy with a construction for the Minkowski function.

Analogous investigations on applications of the Farey fractions to modelling generalizations of the Minkowski function, are given in [51].

So, finally, let us remark that nowadays many researches are devoted to real analysis objects with complicated local structure and related to them problems. Considering connections fractal sets with various measures, singular and non-differentiable functions, as well as mathematical modeling, etc., one can present many surveys according to various types of such relations. For example, other new researches including corresponding auxiliary surveys such that are related to fractals, measures, and their various applications are presented in [52-78]. Considering last-mentioned researches, one can note the following topics of modern investigations in the area of fractals: new multifractal measures and symmetric generalized Cantor sets [52]; a note on level sets and the multifractal analysis for associated Mandelbrot measures [53]; a general vectorial formalism of the vectorial multifractal analysis [54]; applications of recurrent iterated function systems as generalizations of classical iterated function systems [55]; certain generalized affine fractal interpolation functions [56]; fractal properties of product sets and investigations of a premeasure as the Hausdorff function [57]; some types of the regularity of sets and the Hewitt-Stromberg measures for certain cartesian product sets [58]; modeling and applications of graphs of certain continuous functions such that interpolated the given data [59]; investigations of relations between certain premeasures and measures [60]; fractal interpolations including applications of iterated function systems to them [61]; new techniques for defining the multifractal function dimension [62]; the generalized multifractal packing and Hausdorff measures, as well as their connections and certain applications in the set theory [63]; studying certain fractal properties of self-similar sets and some applications of these results to new knowledge on the middle third Cantor set, the Sierpinsky carpet and triangle [64]; various approaches for defining general packing and Hausdorff measures (for example, see [65]); examples of some connections between fractal geometry and approximation theory for a certain expansions of real numbers [66]; studying some connections of probability theory, fractal analysis, and topology [67]; introducing the notions of “the mean packing dimension” and “the mean pseudo-packing dimension” [68]; some fractal properties of a certain homogeneous Cantor sets [69] and Moran sets [70]; properties of some measures which determines “the modified lower box dimension Moran fractal sets” [71]; behavior properties of the multifractal Hewitt-Stromberg dimensions and measures [72,73]; modelling Moran sets satisfying a special property called as “the strong separation condition” [74]; studying “a behavior of the relative multifractal spectrum” [75]; considering multifractal Hausdorff and packing dimensions for a Borel probability measure [76]; Hewitt-Stromberg dimensions and maps [77-78]. In addition, one can note that fixed point results in fuzzy S-metric spaces with applications to fractals and satellite web coupling problems are noted in [79], and modeling fractals in the setting of graphical fuzzy cone metric spaces is presented in [80].

Since singular, non-differentiable, or non-monotone functions as functions with complicated local structure are related with fractals according to dimensions of their graphs or to modelling fractal sets by expansions of arguments or values of such functions, the noted topics of researches are useful for the development the theory of functions with complicated local structure and vice versa.

4. Conclusions

Nowadays many investigations are devoted to generalizations of functions with complicated local structure including singular functions and their connections with fractal analysis (in fractal multiformalism, general fractal measures and dimensions, and Hewitt-Stromberg measures and homogeneous Moran measures, etc.) and with various mathematical areas.

Considering the development of methods and techniques for defining singular functions as functions with complicated local structure, one can remark the methodology of number theory, analytical geometry (when we deal with definitions by constructing of graphs), and probability theory, as well as systems of functional equations. Also, the special attention can be noted to auxiliary techniques of mathematical analysis. Corresponding methods and techniques were introduced by classics of mathematics and their closest followers. This methodology is being continued and extended by a number of researchers since 1940th. For example, such scientists as Weierstrass, Dini, and Takagi, as well as Cantor, Lebesgue, and Minkowski are founders of the theory of functions with complicated local structure. Indeed, as it was noted above, the history of continuous monotone singular functions begins in 1883-1904 with corresponding examples modeled by Cantor in 1884 and also considered by Lebesgue in 1904, by Scheeffer in 1884, and by Vitali in 1905. One of the famous example of singular functions as the one-to-one correspondence between quadratic irrational numbers and rational numbers on the unit interval, was defined by Minkowski in 1904.

The main techniques to construct singular functions can be characterized as following:

In probability theory: using the notions of Markov chains and distribution function;

In number theory: using various expansions of arguments and values of a function, the Stern–Brocot tree and the Farey sequence, etc.;

In geometry: certain geometrical iteration procedures and auxiliary relations for the geometric construction of the graph of a function;

In mathematical analysis: applications of auxiliary maps, compositions of functions, or iterated function systems;

Using functional equations or their systems.

Generative AI Statement

The author declares that no Gen AI was used in the creation of this manuscript.

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