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Approximating Fixed Points of Generalized Cyclic Enriched Contraction Mapping Using Ishi Iteration Scheme with Application

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Abstract

This study shows the presence and uniqueness of the optimal proximity point for several classes of generalized cyclic enriched contractions, and offers such fundamental results. We provide convergence results for this contraction. We also provide the conditions in which an iterative method can yield the optimal proximity point. To further illustrate the effectiveness of the ishi technique for generalized cyclic enriched contractions, we present a numerical solutions with comparison table and grapshical analysis, which show that our proposed iterative scheme converges faster than the other schemes. Our results are generalization of many comparable results in literature. In addition, the theoretical framework developed in this study extends classical fixed point and best proximity point results by relaxing standard contraction assumptions. The proposed approach allows a broader class of mappings to be analyzed within a unified setting. The convergence analysis is supported by rigorous proofs, ensuring the reliability of the proposed iterative method. Moreover, the numerical experiments validate the theoretical findings and demonstrate the stability and efficiency of the method under different initial conditions. The comparison with existing iterative schemes highlights the superiority of the proposed algorithm in terms of convergence speed and accuracy. These results indicate that the ishi technique is a powerful and flexible tool for solving proximity point problems arising in nonlinear analysis. Consequently, the findings of this study contribute meaningfully to the existing literature and open new directions for further research in generalized contraction mappings and iterative approximation methods.

Keywords

Best proximity point, Bianchini contraction, Fixed point, Generalized cyclic enriched contraction, Ishi algorithm

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1. Introduction

Distance serves as a fundamental framework for accurate estimation and measurement across various fields. It enables us to quantify and compare physical attributes such as lengths, heights, widths, and depths. Additionally, distance helps determine the size, parameters, and proportions of objects. By clearly defining measurement or distance operations, mathematicians can establish the foundation for calculations, comparisons, and analyses in diverse mathematical areas.

In 1922, the study of mappings and their fixed points led to significant discoveries, such as the Banach Contraction Principle (BCP) [1], which confirms the existence and stability of fixed points in complete metric spaces. Fixed points are unique in that the distance between a point and its image under a function is zero. Mathematical theorems often rely on distance-related properties, such as contraction or nonexpansive mappings, to ensure that the distance between points either decreases or remains unchanged.

Additionally, the BCP introduces a method called Picard iteration [2] for finding the unique fixed point of a contraction mapping. Research articles using fixed-point theorems often explore the practical applications of different interpretations of the Banach principle. If a function is nonexpansive (meaning it doesn’t increase distances) within its domain, it may still have a fixed point under certain conditions. However, the limit of the Picard iteration may not always yield a fixed point for nonexpansive mappings. To improve accuracy and efficiency, researchers have developed generalized iterative methods that outperform traditional approaches.

The generalized cyclic enriched contraction framework is motivated by the need to model iterative processes that naturally evolve across multiple subsets rather than within a single domain. From a geometric viewpoint, cyclic enrichment relaxes the uniform contraction requirement by allowing controlled movement between subsets, which enlarges the region of convergence while preserving stability. This structure reflects many practical iterative dynamics and provides greater flexibility in ensuring convergence compared to classical contraction mappings.

The theory of best proximity points is closely tied to fixed-point theory, which is a powerful mathematical tool. Studying best proximity points helps deepen our understanding of fixed-point concepts and provides solutions to various mathematical challenges. It also plays a significant role in optimization problems, especially when identifying solutions that are farthest from a given set or point. This is particularly important in fields like engineering, economics, and operations research. Inspired by Brinde et al.’s work on enriched contraction mappings [3,4] and Karpinar’s generalized cyclic contraction framework [5], this article introduces a new class of mappings called "generalized cyclic enriched contraction mappings," which unifies and extends existing generalized cyclic contraction concepts. The resulting maps are a valuable tool for determining the ideal proximity points. The best proximity feature outcomes for generalized cyclic improved contraction mappings can be applied to cyclic mappings as well enhanced Kannan contracture mappings. We expand the cyclic reinforced Kannan contraction maps to include cyclic enriched Bianchini contraction mapping.

Let (Ω, d) be a complete metric space, and let $T, I \subseteq \Omega$ be nonempty closed subsets. The distance between T and I is defined as: $Dist(T,I) = \inf\{d(l, j) : l \subset T, j \subset I\}$.

We define the sets of points in T and I attaining this minimal distance as:

$T\_0 = \{l \subset T : d(l, j) = Dist(T,I) \text{ for some } j \subset I\}$,

$I\_0 = \{j \subset I : d(l, j) = Dist(T,I) \text{ for some } l \subset T\}$.

Fixed points are accurate points that are assigned to themselves [2]. Best proximity values are those that are closest to specific components or sets, even if they fail to coincide entirely. The best proximity point theory expands on the idea of fixed points to include mappings without fixed points [6]. The best proximity points can be seen as approximate mapping possibilities [7-9].

Suppose that a mapping $\lambda: T \cup I \rightarrow T \cup I$ is cyclic with respect to (T, I), i.e., $\lambda(T) \subseteq I$ and $\lambda(I) \subseteq T$.

A point $\omega \in T \cup I$ is called a best proximity point of λ, if $d(\omega, \lambda\omega) = Dist(T,I)$.

2. Preliminaries

Throughout this manuscript, (Ω, d) denotes a metric space. When Ω is equipped with a norm $\|\cdot\|$, it is assumed to be a Banach space. The symbols T and I denote nonempty subsets of Ω.

$Dist(T,I) = \inf\{d(i, j): i \in T, j \in I\}$.

Elements of Ω are denoted by i, j, k. The mapping $\lambda: T \cup I \rightarrow T \cup I$ represents a cyclic-type operator, and λ_σ denotes its enriched form defined via a convex structure ∇. The constant l denotes a contraction parameter and $\sigma \in [0,1)$ is a control parameter.

Definition 1 [10,11]. A mapping λ: T ∪ I→T ∪ I is called a cyclic contraction. If, $\lambda(T) \subseteq I$ and $\lambda(I) \subseteq T$; for every i ∈ T and j ∈ I, there exists l ∈ (0,1), such that, $d(\lambda i, \lambda j) \leq l\, d(i, j) + (1-l)Dist(T,I)$.

Definition 2 [12]. A Banach space Ω is said to be uniformly convex Banach (UCB) space if there exists an increasing function δ: (0,2]→[0,1], such that, for all i, j, k ∈ Ω, and for some K > 0 and κ ∈ [0, 2K],

$$\begin{cases} \|i-k\| \leq K \\ \|j-k\| \leq K \\ \|i-j\| \geq \kappa \end{cases} \Rightarrow \|\frac{(i+j)}{2}-K\| \leq (1-\delta(\frac{\kappa}{K}))K.$$

Lemma 1 [13]. Let T and I be nonempty closed convex subsets of a UCB space Ω.

Suppose there exist sequences $\{i_n\}$, $\{k_n\} \in T$, $\{j_n\} \in I$, such that $\|k_n-j_n\| \rightarrow Dist(T, I)$, for every ε > 0, ∃ $N_0 \in N$, such that ∀ $m > n \geq N_0$, $\|i\_m - j\_n\| \leq Dist(T,I) + \varepsilon$, then, for every ε > 0 and all $m > n \geq N_0$,

$\|i_m-k_n\| \leq \varepsilon$.

Lemma 2 [13]. Let T be a nonempty closed convex subset and I a nonempty closed subset of a UCB space Ω.

Assume there exist sequences $\{i_n\}$, $\{k_n\} \in T$, $\{j_n\} \in I$, such that, $\|i_n-j_n\| \rightarrow Dist(T,I)$, $\|k_n-j_n\| \rightarrow Dist(T,I)$, then, $\|i_n-k_n\| \rightarrow 0$.

Definition 3. Let (Ω, d) be a metric space.

A mapping, ∇: Ω × Ω × [0,1]→Ω, is called a convex structure,

if for all i, j, k ∈ Ω and σ ∈ [0,1],

$d(k, \nabla(i, j; \sigma)) \leq \sigma\, d(k, i) + (1 - \sigma)\, d(k, j)$.

The triplet (Ω, d, ∇) is called a convex metric space.

Definition 4. Let (Ω, d, ∇) be a convex metric space.

A nonempty subset T ⊆ Ω is said to be convex if for all i, j ∈ T and σ ∈ [0,1], ∇(i, j; σ) ∈ T.

Lemma 3 [14]. Let (Ω, d, ∇) be a convex metric space.

For all i, j ∈ Ω and $\sigma, \sigma_1, \sigma_2 \in [0,1]$, the following properties hold:

$\nabla(i, i; \sigma) = i$, $\nabla(i, j; 0) = j$ and $\nabla(i, j; 1) = i$

$D(i, \nabla(i, j; \sigma)) = (1 - \sigma)d(i, j)$ and $d(j, \nabla(i, j; \sigma)) = \sigma\, d(i, j)$

$d(i, j) = d(i, \nabla(i, j; \sigma)) + d(\nabla(i, j; \sigma), j)$

$|\sigma_1-\sigma_2|\, d(i, j) \leq d(\nabla(i, j; \sigma_1), \nabla(i, j; \sigma_2))$.

For a self-map, λ: Ω→Ω, the fixed point set is defined by Fix(λ) = {i ∈ Ω: λ i = i}.

This lemma generalizes Broder and Petrshyn's [15] conclusion via Banach to convex metric spaces.

Lemma 4. Let (Ω, d, ∇) be a convex metric space and let λ: Ω→Ω.

For σ ∈ [0,1), define λ_σ: Ω→Ω by $\lambda_\sigma(i) = \nabla(i, \lambda i; \sigma)$, then $Fix(\lambda) = Fix(\lambda_\sigma)$.

Berinde [4] introduced enriched non-expansive mappings in 2019, extending classical fixed point theory. This enrichment strengthened both theoretical foundations and applications in nonlinear modeling. Subsequently, Berinde, Păcurar, and others [16-19] proposed improved contractive mappings in convex metric spaces, unifying Banach contractions and non-expansive mappings. Krasnoselskii-type iterations were shown to approximate unique fixed points effectively [20], highlighting the importance of iterative schemes in analysis and optimization. Further studies focused on stability and convergence of iterative algorithms. He [21] established convergence results for Gauss-type proximal point methods under metric regularity. Adamu et al. [22,23] developed Tseng-type algorithms for monotone and variational inclusion problems, proving strong convergence in Banach spaces. These results demonstrate the efficiency of relaxed and enriched iterative methods. Following work extended enriched contractions to cyclic and proximity settings. Abbas et al. [24] and De la Sen [25] studied generalized enriched cyclic contractions and their stability properties. Dechboon and Khammahawong [26] obtained best proximity point results with convergent algorithms.

Definition 5 (Enriched Contraction). Suppose that convex metric space (Ω, d, ∇) and a mapping from itself λ: Ω→Ω, now define another mapping λσ: Ω→Ω by $\lambda_\sigma(i) = \nabla(i, \lambda_\sigma; \sigma)$,

for every i ∈ Ω, if there lies l and σ ∈ [0,1), such that $d(\lambda_\sigma i, \lambda_\sigma j)$=$d(\nabla(i, \lambda_\sigma;\sigma)), d(\nabla(j, \lambda_j;\sigma)) \leq l\, d(i,j)$.

This map is then referred to as an enriched contraction.

Definition 6 (Cyclic Enriched Contraction). A map λ: T ∪ I→T ∪ I with non-empty subsets T and I of convex metric space then it is called cyclic enriched contraction map if it satisfies the following:

λ(T) ⊂ I, λ(I) ⊂ T and for all i ⊂ T, j ⊂ I, that is

$$d(\lambda_\sigma\, i, \lambda_\sigma\, j) = d(\nabla\ (i, \lambda_{\sigma;\sigma})), d(\nabla\ (j, \lambda_{j;\sigma})) \le l\, d(i, j) + (1 - l)\, Dist(T,I) \quad (1)$$

Definition 7 (Generalized Cyclic Enriched Contraction) [26]. A map λ: T ∪ I→T ∪ I has non empty subsets T and U of convex metric space then it is considered as generalized cyclic enriched contraction map if it satisfies the following requirements:

λ(T) ⊂ I, λ(I) ⊂ T

for all i ⊂ T and j ⊂ I, there is l ⊂ $\left[0,\frac{1}{3}\right]$ and γ∈[0,∞], that is

$$d(\lambda_\sigma\, i, \lambda_\sigma\, j) = d(\nabla\ (i, \lambda_\sigma;\sigma)), d(\nabla\ (j, \lambda_j;\sigma)) \le l\, d(i, j) + d(i,\lambda_\sigma\, i) + d(j,\lambda_\sigma\, j)+(1 - l)\, Dist(T,I) \quad (2)$$

Definition 8 (Generalized Cyclic (γ, κ) Enriched Contraction) [26]. A map λ: T ∩ I→T ∩ I has non empty subsets T and I of convex metric space then it is considered as generalized cyclic enriched contraction map if it satisfies the following requirements:

λ(T) ⊂ I, λ(I) ⊂ T

for all i ⊂ T and j ⊂ I, there is l ∈ $\left[0,\frac{1}{3}\right]$ and γ∈[0,∞], that is

$$\left\|\gamma(i\text{-}j)+\lambda i\text{-}\lambda j\right\| \le l\left(\left\|i\text{-}j\right\| + \left\|i\text{-}\lambda i\right\| + \left\|j\text{-}\lambda j\right\|\right) + (1\text{-}3l)Dist(T,I) \quad (3)$$

it is being observed that Equation (2) is generalized enriched for $\sigma=\frac{1}{\lambda+1}$ and by getting value of σ = 1 we put λ = 0,

$\left\|\lambda i\text{-}\lambda j\right\| \le l\left(\left\|i\text{-}j\right\| + \left\|i\text{-}\lambda i\right\| + \left\|j\text{-}\lambda j\right\|\right) + (1\text{-}3l)Dist(T,I)$.

If we take γ > 0, then

$\left\|\left(\frac{1}{\sigma}\text{-}1\right)(i\text{-}j) + \lambda i\text{-}\lambda j\right\| \le l\left(\left\|i\text{-}j\right\| + \left\|i\text{-}\lambda i\right\| + \left\|j\text{-}\lambda j\right\|\right) + (1\text{-}3l)Dist(T,I)$.

Finally, the result is

$$\|(\lambda_\sigma i, \lambda_\sigma j)\| < l\left\|i\text{-}j\right\| + \|i\text{-}\lambda_\sigma i\| + \|j\text{-}\lambda_\sigma j\| + (1\text{-}3l)\, Dist(T,I) \quad (4)$$

Definition 9 (Generalized Cyclic Kannan Enriched Contraction) [26,27]. A map λ: T ∩ I→T ∩ I has non empty subsets T and I of convex metric space then it is considered as generalized cyclic Kannan (γ, k) enriched contraction map if it satisfies the following requirements:

λ(T) ⊂ I, λ(I) ⊂ T

for all i ⊂ T and j ⊂ I,

there is l ∈ $\left[0,\frac{1}{2}\right]$ and γ∈[0,1], that is

$d(\lambda_\sigma i, \lambda_\sigma j)=d(\nabla(i,\lambda_i;\sigma)),d(\nabla(j,\lambda_j;\sigma))\ l\ d(i, \lambda_\sigma i)+d(j, \lambda_\sigma j)+(1\text{-}2l)\ Dist(T,I)$.

Lemma 5. Chndok [4] investigated the convergence process and optimum approximation for cyclic-enriched contraction maps.

Assume Ω is a UCBS with non-empty subsets T and I.

Let λ: T ∪ I→T ∪ I, such that λ(T) ⊆ I, λ(I) ⊆ T, for i_0 ∈ T, define the iteration, $i_{n+1} = \lambda_\sigma i_n = 1\text{-}_\sigma\lambda i_n+\sigma i_n$.

Non-expansive mappings were further discussed by Tehreem et al. [28]. Proximity principles also appeared in applied contexts [29,30]. Finally, Chhatrajit [31] extended best proximity point theory to multiplicative metric spaces. This body of work provides the theoretical basis and motivation for the enriched mapping framework and iterative algorithm developed in the next section, where new convergence results are established.

3. Examples

3.1 Example 1

(1) It is immediate from Lemma 1 that every generalized cyclic contraction mapping is a particular case of a generalized cyclic (0, l)-enriched contraction mapping by choosing γ = 0.

(2) Let Ω = ℝ endowed with the usual norm. Define two nonempty closed subsets,

T = [0, 1], I = [-2, -1]

and define the mapping λ: T ∪ I⟶T ∪ I by λ(i) = {-i-1, i ∈ T; -i-1, i ∈ I}.

Hypotheses:

Clearly, λ(T) ⊂ I and λ(I) ⊂ T, hence λ is a cyclic mapping. Moreover,

Dist(T,I) = inf{|x-y|:x ∈ T, y ∈ I} = 1

For any i,j ∈ T ∪ I, we have |λi-λj| = |i-j|, which shows that λ is nonexpansive and therefore not a generalized cyclic contraction mapping.

Enriched contractive condition.

Let $\gamma \in (0,\frac{1}{3})$ and set γ = 1-3l ∈ [0,1). Then, for all I ∈ T and j ∈ I,

$|\gamma(i-j) + \lambda i-\lambda j| = |(\gamma-1)(i-j)| = 3l|i-j| \leq l(|i-j|+|2i+1| + |2j+1|)$.

Adding (1−3l)Dist(T,I) on both sides, we obtain

$|\gamma(i-j) + \lambda i-\lambda j| \leq l(\cdot) + (1-3l)\text{Dist}(T,I)$

which confirms that λ is a generalized cyclic (1−3l,l)-enriched contraction mapping.

Since T ∩ I = ∅ and Dist(T,I) = 1, the optimal proximity point is attained at $I^* = 0 \in T$ and $\lambda(I^*) = -1 \in I$.

3.2 Example 2

(1) It is immediate from definition that every generalized cyclic contraction mapping is a particular case of a generalized cyclic (0,l)-enriched contraction mapping by choosing γ = 0.

(2) Let Ω = ℝ endowed with the usual norm.

Define two nonempty closed subsets, T = [1,2], $I = [\frac{-3}{2},\frac{-1}{2}]$ and define the mapping λ: T ∪ I⟶T ∪ I by

$$\lambda(i)=\begin{cases}-i+\frac{1}{2} \\ -i+\frac{1}{2}\end{cases} \quad i \subset T \text{ and } I.$$

Hypotheses:

Clearly, λ(T) ⊂ I and λ(I) ⊂ T, hence λ is a cyclic mapping. Moreover, $\text{Dist}(T,I) = \frac{3}{2}$, which shows that λ is nonexpansive and therefore not a generalized cyclic contraction mapping.

3.2.1 Enriched Bianchini Condition

Let $l \in (0,\frac{1}{2})$ and set γ = 1-2l ∈ [0,1). Then, for all I ∈ T and j ∈ I,

$|\gamma(i-j)+\lambda i-\lambda j|=|(\gamma-1)(i-j)|=2l|i-j|\leq l|2i-\frac{1}{2}|+|2j-\frac{1}{2}|$

Adding (1−2l)Dist(T,I) on both sides, we obtain

$\big||\gamma(i-j)+\lambda i-\lambda j|\big|+2l\text{Dist}(T;I)\leq 2l\max(|2i-\frac{1}{2}|,|2j-\frac{1}{2}|+3l\ \text{Dist}(T,I)$

$<2l\max(|2i-\frac{1}{2}|, |2j-\frac{1}{2}|+ \text{Dist}(T,I)$

$|\gamma(i-j)+\lambda i-\lambda j|<2l\max(|2i-\frac{1}{2}|, \left|2j-\frac{1}{2}\right| +(1-2l)\text{Dist}(T,I).$

Which confirms that λ is a generalized cyclic (1−2l,)-enriched Bianchini contraction mapping.

3.2.2 Optimal Proximity Point

Since T ∩ I = ∅ , the optimal proximity point is achieved at $I^* = 1 \in T$ with

$|I^*-\lambda(I^*)| = \text{Dist}(T,I)$.

3.3 Example 3

We offer a mathematical test to demonstrate the convergence of the ishi method.

$t_n = \lambda[(1\text{-}\sigma)i_n+\sigma\lambda i_n]$, $X_n = \lambda t_n$, $i_{n+1} = (1\text{-}\sigma)\lambda i_n+\sigma\lambda i_n$.

In the category of generalized enhanced contraction mappings. Consider T = [0,6] with the normal norm and a map λ: T→T defined by $\lambda i = \frac{i+15}{2}$ has initial points i_0 = 2.2 and 4.4 and took different values of σ.

A mapping λ is referred to as:

(1) Banach contraction mapping then there lies some $l \in [0,1)$ so that $|\lambda i-\lambda j| \leq l|i-j|$.

(2) Generalized form is $|\lambda i-\lambda j| \leq l(|i-j|+|i-\lambda i|+|j-\lambda j|)$.

Simply when we choose i = 2, j = 0 then the equation yields $l \leq 2l$ so it is clear λ is not a Banach contraction.

Simply when we choose i=2, j=0 then the equation yields $l \leq 6l$ so it is clear λ is not generalized contraction, then it contradicts that $l \leq 1$

The speed-up of our ishi algorithm in Example 3 is shown in Table 1. Graphical analysis of our scheme for table 1 is shown in Figure 1.

Table 1. The speed-up of our ishi algorithm in Example 3.

n	σ = 0.1	σ = 0.3	σ = 0.5	σ = 0.7	σ = 0.9
0	2.2	2.2	2.2	4.4	4.4
1	0.557064	0.556747	0.556467	0.556600	0.556277
2	0.555546	0.555446	0.555556	0.555555	0.555555
3	0.555554	0.555554	0.555555	0.555555	0.555555
4	0.555555	0.555555	0.555555	0.555555	0.555555
5	0.555555	0.555555	0.555555	0.555555	0.555555
6	0.555555	0.555555	0.555555	0.555555	0.555555
7	0.555555	0.555555	0.555555	0.555555	0.555555
8	0.555555	0.555555	0.555555	0.555555	0.555555
9	0.555555	0.555555	0.555555	0.555555	0.555555
10	0.555555	0.555555	0.555555	0.555555	0.555555
11	0.555555	0.555555	0.555555	0.555555	0.555555
12	0.555555	0.555555	0.555555	0.555555	0.555555

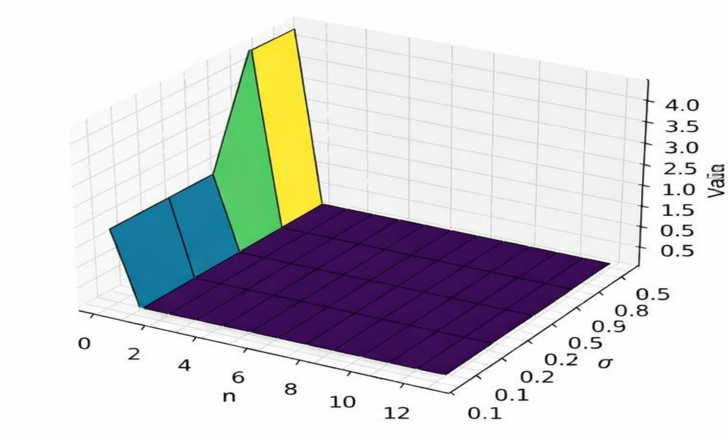

Figure 1. Graphical analysis of our scheme for table 1.

4. Main Results

This article explains the criteria for the existent Fixed value for generalized cyclic enriched contraction mapping in frequent convex Banach space.

Theorem 1. Suppose that convex metric space (Ω, d, ∇) and mapping λ: Ω→Ω. has non empty subsets T and U of metric space Ω then it is considered as generalized cyclic enriched contraction map. There is any initial point $i_0 \in T \cup I$, so we obtain $d(i_n,\lambda_\sigma i_n) \to Dist(T,I)$ where $i_{n+1}= \nabla(i_n,\lambda_\sigma i_n;\sigma)=\lambda_\sigma i_n$, $n \geq 0$.

Proof:

By the definition of generalized cyclic enriched contraction, we have

$d(i_1,i_2) = d(\lambda_\sigma i_0,\lambda_\sigma i_1) \leq l d(i_0,i_1) + d(i_0,\lambda_\sigma i_0) + d(i_1,\lambda_\sigma i_1) + (1 - 3l)(T,I)$.

Using the identities $d(i_0,\lambda_\sigma i_0) = d(i_0,i_1)$ and $d(i_1,\lambda_\sigma i_1) = d(i_1,i_2)$, we obtain

$d(i_1,i_2) \leq \; ld(i_0,i_1) + d(i_0,i_1) + d(i_1,i_2) + (1 - 3l)(T,I)$.

Rearranging yields,

$$d(i_1,i_2) \leq \frac{2l}{1-l} d(i_o,i_1) + \frac{1-3l}{1-l}(T,I) \quad (5)$$

Applying the same argument to i_1 and i_2, we obtain

$$d(i_2,i_3) \leq \frac{2l}{1-l} d(i_1,i_2) + \frac{1-3l}{1-l}(T,I) \quad (6)$$

Substituting Equation (5) into Equation (6) gives

$d(i_2,i_3) \leq \left(\frac{2l}{1-l}\right)^2 d(i_0,i_1) + \left(\frac{1}{1-l} + \frac{2l}{(1-l)^2}\right) + (1-3l)(T,I)$.

Proceeding inductively, we obtain

$d(i_n,i_{n+1}) \leq \left(\frac{2l}{1-l}\right)^2 d(i_0,i_1) + \left(\frac{1}{1-l} + \frac{2l}{(1-l)^2} + \ldots + \frac{(2l)^{n-1}}{(1-l)^n}\right) + (1-3l)(T,I)$

Since $l \in [0,\frac{1}{3}]$, we have $\frac{2l}{1-l} \in [0,1]$. Taking the limit as $n \to \infty$ yields

$\lim d(i_n,i_{n+1}) = Dist(T,I)$.

Theorem 2. Suppose that convex metric space (Ω, d, ∇) and mapping $\lambda: \Omega \to \Omega$. has non empty subsets T and U of metric space Ω, then it is considered as generalized cyclic enriched contraction map, then there is any initial point $i_0 \in T \cup I$, so we obtain $d(i_n,\lambda_\sigma i_n) \longrightarrow Dist(T,I)$, where $i_{n+1} = \nabla(i_n,\lambda i_n;\sigma) = \lambda_\sigma i_n$, $n \geq 0$. if $\{i_{2n}\}$ has convergent subsequences in T then there lies an element ω for this $d(\omega,\lambda_\sigma\omega) = Dist(T,I)$.

Proof:

Let $\{i_{2nk}\}$ be a convergent subsequence of $\{i_{2n}\}$, such that

$i_{2nk} \to \omega \in T$.

Using the contractive condition, we have

$d(i_{2nk}+1,\lambda_\sigma\omega) = d(\lambda_\sigma i_{2nk},\lambda_\sigma\omega) \leq ld(i_{2nk},\omega) + d(i_{2nk},\lambda_\sigma i_{2nk}) + d(\omega,\lambda_\sigma\omega) + (1 - 3l)(T,I)$.

By Theorem 1,

$\lim d(i_{2nk},\lambda_\sigma i_{2nk}) = (T,I)$. $k \to \infty$

Taking limits as $k \to \infty$ in the above inequality yields $(T,I) \leq d(\omega,\lambda_\sigma\omega) \leq (T,I)$,

and hence $d(\omega,\lambda_\sigma\omega) = (T,I)$.

Theorem 3. Suppose that (Ω,d,∇) is convex metric space and a mapping $\lambda: \Omega \to \Omega$. is a generalized cyclic (γ,l) enriched contraction, where T and I are nonempty subsets of Ω and T is closed.

Let $i_{n+1} = \lambda_\sigma i_n$, $\sigma = \frac{1}{\gamma+1}$.

If $(T,I) = 0$, then λ admits a fixed point $\omega \in T \cap I$.

Proof:

By Theorem 1, $\lim d(i_n,i_{n+1}) = 0$, as $n \to \infty$, which implies that the sequence $\{i_{2n}\}$ is Cauchy. Since Ω is complete and T is closed, there exists $\omega \in T$, such that, $i_{2n} \to \omega$.

By Theorem 2 and the assumption T, I = 0, we obtain $d(\omega,\lambda_\sigma\omega) = 0$, and hence $\lambda_\sigma\omega = \omega$. Therefore, $\omega \in T \cap I$ is a fixed point of λ.

Theorem 4. Suppose that convex metric space (Ω,d,∇) and a mapping $\lambda: \Omega \to \Omega$ has non empty subsets T and U of metric space Ω and T is closed then it is considered as generalized cyclic (γ,l) enriched contraction map, then there is best proximity point $\omega \in T$, in addition if $i_o \in T$, and $i_{n+1=} \lambda_\sigma i_n = (1 - \sigma)\lambda i_n + \sigma\lambda i_n = \lambda_\sigma i_n$ has $\sigma = \frac{1}{\gamma+1}$. Then the sequence i_{2n} possess best proximity point ω.

Proof:

It is obvious that λ is generalized (γ, l) enriched contraction mapping. There lies $\gamma \in [0,\infty)$ with $l \in \left[0,\frac{1}{2}\right]$, so that,

$\|\gamma(i - j) + \lambda i - \lambda j\| \leq l(\|i - j\| + \|i - \lambda i\| + \|j - \lambda j\|) + (1 - 3l)Dist(T,I)$.

For the value $\gamma=0$, here comes $l = 1$, we obtain

$\|\lambda i - \lambda j\| \leq l(\|i - j\| + \|i - \lambda i\| + \|j - \lambda j\|) + (1 - 3l)Dist(T,I)$.

The following output are generated from [28] For $\gamma>0$, and $\sigma=\frac{1}{\gamma+1}$.

Then, we observe

$\|\lambda_\sigma i - \lambda_\sigma j\| \leq l(\sigma\|i - j\| + \|i - \lambda_\sigma i\| + \|j - \lambda_\sigma j\|) + \sigma(1 - 3l)Dist(T,I) < l(\|i - j\| + \|i - \lambda_\sigma i\| + \|j - \lambda_\sigma j\|) + (1 - 3l)Dist(T,I)$

Assume that Dist(T,I) = 0,

Then by utilizing Theorem 1, we get $\lim \|i_{2n} - \lambda_\sigma i_{2n}\| = Dist(T,I)$. as $n\longrightarrow\infty$.

Also there is some N_o and for every $m>n>N_0$,

$\|i_{2m} - \lambda_\sigma i_{2n}\| < Dist(T,I) + \epsilon$.

By observing above inequalities we deduced that the sequence is cauchy and its converges thus it implies that it possess best proximity poiny.

Now it is being proved that best proximity point is unique and for this we take two different points p and q. Thus $\|q - \lambda_\sigma q\| = Dist(T,I)$ and $\|p - \lambda_\sigma p\| = Dist(T,I)$.

Now begin with hypotheses that result in conflicts.

$\|p - \lambda_\sigma q\| \leq \|p - i_{2n+1}\| + \|i_{2n+1} - \lambda_\sigma q\| < \|p - i_{2n+1}\| + l\|i_{2n} - q\| + \|i_{2n} - \lambda_\sigma i_{2n}\| + \|q - \lambda_\sigma q\| + (1 - 3l)Dist(T,I)$.

By rearrange, we obtain

$\left|\|p-\lambda_\sigma q\|\right| \leq \frac{1+l}{1-l}\left|\|p-i_{2n+1}\|\right| + \frac{2l}{1-l}\left|\|p-i_{2n}\|\right| + \frac{2l}{1-l}\left|\|q-\lambda_\sigma q\|\right| + \frac{1-3l}{1-l}Dist(T,I)=Dist(T,I).$

Let $n \longrightarrow \infty$, we obtain

$\left|\|p-\lambda_\sigma q\|\right| \leq + \frac{2l}{1-l}Dist(T,I) + \frac{1-3l}{1-l}Dist(T,I)=Dist(T,I).$

Thus $\|p - \lambda_\sigma q\| = Dist(T,I)$, so it is as

$\left|\left|\frac{p+q}{2}-\lambda_\sigma q\right|\right|^2=\left\|\frac{p-\lambda_\sigma q}{2}+\frac{q-\lambda_\sigma q}{2}\right\|=\frac{1}{2}\|p-\lambda_\sigma q\|^2+\frac{1}{2}\|q-\lambda_\sigma q\|^2-\frac{1}{4}\|p-q\|^2$

We may modify it as:

$\|p-q\|^2=2\|p-\lambda_\sigma q\|^2+2\|q-\lambda_\sigma q\|^2-4\left|\left|\frac{p+q}{2}-\lambda_\sigma q\right|\right|^2.$

Now by using definition of Dist(T,I), we get

$\|p - q\|^2 = 2Dist(T,I)^2 + 2Dist(T,I)^2 - 4Dist(T,I)^2 = 0$.

Hence it states that p=q, so uniqueness of best proximity is verified.

When T = I for Theorem 4, we can conclude that self-map has existence of Fixed point.

Theorem 5. Suppose that T as convex closed subset of metric space (Ω, d, ∇) and a mapping $\lambda: \Omega\longrightarrow\Omega$ is considered as generalized enriched contraction map, then there is,

$Fix(\lambda) = \omega$ and some $\omega \in T$,

sequence modified by ishi method,

$i_{n+1} = \nabla(i_n,\lambda i_n;\sigma) = \lambda_\sigma i_n$

possess a convergence point ω, let for any $i_o \in T$.

Proof:

Let it is already done that λ is generalization of enriched map, we consider another map $\lambda_\sigma i = \nabla(i_n,\lambda i_n;\sigma)$ satisfies

$d(\lambda_\sigma i,\lambda_\sigma j) \leq l(d(i,j) + d(i,\lambda_\sigma i) + d(j,\lambda_\sigma j))$,

let consider $i = i_n$ and $j = j_{n-1}$ and by observing ishi algorithm, we deduce

$d(i_{n+1},i_n)\leq\left(\frac{2l}{1-l}\right)^2 d(i_0,i_1)$, $n \geq 1$.

For $m > n$ and applying $\lim_{n\longrightarrow\infty} d(i_{n+1},i_n) = 0$, we obtain

$d(i_n,i_m)\leq d(i_n,i_{n+1})+d(i_{n+1},i_{n+2})+\ldots+d(i_{m-1},i_m) \leq\left(\left(\left(\frac{2l}{1-l}\right)^n+\left(\frac{2l}{1-l}\right)^{n+1}+\ldots+\left(\frac{2l}{1-l}\right)^{m-1}\right) d=\frac{\left(\frac{2l}{1-l}\right)^n\left(1-\left(\frac{2l}{1-l}\right)^{m-n}\right)}{1-\frac{2l}{1-l}} d(i_0,i_1).\right.$

Applying $\lim_{n\to\infty} d(i_n,i_m) = 0$, so the sequence is cauchy and it states that it is convergent.

$\lim i_n = \omega$. as $n\to\infty$.

Now utilising λ is generalized enriched contraction mapping so

$d(\omega,\lambda_\sigma\omega) \leq d(\omega,i_{n+1}) + d(i_{n+1},\lambda_\sigma\omega) \leq d(\omega,i_{n+1}) + l(d(i_n,\omega) + d(i_n,i_{n+1}) + d(\omega,\lambda_\sigma\omega)$.

By applying the limit, we obtain $\lambda_\sigma\omega = \omega$. So $0 = d(\omega,\lambda_\sigma\omega) = d(\omega,\nabla(i_n,\lambda i_n;\sigma) = (1-\sigma)\lambda_\sigma\omega \Longrightarrow d(\omega,\lambda_\sigma\omega)$.

Corollary 1. Suppose that T as convex closed subset of metric space (Ω, d, ∇) and a map λ: $T \longrightarrow T$ is considered as generalized (γ,l) enriched contraction map, then there is,

$Fix(\lambda) = \omega$ and some $\omega \in T$,

there is $\sigma \in (0,1]$ so the sequence modified by ishi method

$i_{n+1} = (1-\sigma)\lambda i_n + \sigma\lambda i_n$

possess a convergence point ω, let for any $i_o \in T$.

Proof:

By putting value of $\sigma = \frac{1}{\gamma+1}$, we obtain our desired outcome.

Example 4. Consider the convex Banach space X = R with the Euclidean norm. Let F = G = [0,1], then F and G are nonempty, closed, and convex subsets of X and Dist(F,G) = 0. Define the mapping λ: $T\to T$ by $\lambda i = 1 - i$, $\forall i \in T$.

Indeed, λ is a generalized enriched contraction mapping. There exist $b \in [0,\infty)$ and $k \in \left[0,\frac{1}{3}\right]$, such that

inequality (3) holds, which is equivalent to

$$|(b-1)(i-j)\leq k(|i-j|+|2i-1|+|2j-1|),\ \forall\ i,j\in[0,1] \tag{7}$$

Since

$3|i-j|\leq(|i-j|+|2i-1|+|2j-1|),\ \forall\ i,j\in[0,1]$

we observe that Equation (7) is valid for any $k \in \left[0,\frac{1}{3}\right]$. If we set $b = 1-3k$ (thus, $0 < b \leq 1$), hence, for any $k \in \left[0,\frac{1}{3}\right]$ is a generalized $(1-3k,k)$-enriched mapping and $Fix(j)=\frac{1}{2}$.

Therefore, we can conclude that the Corollary holds, as all the conditions have been satisfied.

Cyclic Enriched Bianchini Contraction Map

Definition 10. A map λ: $T \cup I \longrightarrow T \cup I$ has non empty subsets T and U of convex metric space then it is considered as generalized cyclic Bianchini contraction enriched map if it satisfies the following requirements:

$\lambda(T) \subseteq I$ and $\lambda(I) \subseteq T$;

for every $i \in T$ and $j \in I$ there lies τ and $\sigma \in [0,1)$, that is

$$d(\lambda_\sigma i,\lambda_\sigma j) = d(\nabla(i,\lambda i;\sigma)),d(\nabla(j,\lambda j;\sigma)) \leq \tau\max(d(i,\lambda_\sigma i),d(j,\lambda_\sigma j)) + (1-\tau)Dist(T,I) \tag{8}$$

Theorem 6. A convex closed subset metric space (Ω, d, ∇) with a map λ: $T\cup I \longrightarrow T\cup I$ has non empty subsets T and U of convex metric space then it is considered as generalized cyclic Bianchini contraction enriched map then any initial value $i_o \in T \cup I$, we get $d(i_n,\lambda_\sigma i_n)) \longrightarrow Dist(T,I)$, where $i_{n+1} = \nabla(i_n,\lambda i_n;\sigma) = \lambda_\sigma i_n$.

Proof:

Here the definition of genralized cyclic Bianchini contracation enriched mapping is being utilised.

$d(i_1,i_2) = d(\lambda_\sigma i_o,\lambda_\sigma i_1) \leq \tau[\max d(i_o,\lambda_\sigma i_o),d(i_1,\lambda_\sigma i_1)] + (1-\tau)Dist(T,I) = \tau[\max d(i_o,i_1),d(i_1,i_2)] + (1-\tau)Dist(T,I)$.

Now in case of max the Dist(T,I) is feasible to acquire. If $Dist(T,I) < \max d(i_o,i_1),d(i_1,i_2)] = d(i_1,i_2)$, then

$d(i_1,i_2) \leq \tau d(i_1,i_2) + (1-\tau)Dist(T,I) < d(i_1,i_2)$.

But this shows a contradiction, so, $d(i_1,i_2) \leq \tau d(i_o,i_1) + (1-\tau)Dist(T,I)$, and $d(i_2,i_3) \leq \tau d(i_1,i_2) + (1-\tau)Dist(T,I)$.

Using together above two equations, we obtain

$d(i_2,i_3) \leq \tau(\tau d(i_o,i_1) + (1-\tau)Dist(T,I) + (1-\tau)Dist(T,I) = \tau^2 d(i_o,i_1) + (1-\tau^2)Dist(T,I)$.

So, we have

$d(i_n,i_{n+1}) \leq \tau^n d(i_o,i_1) + (1-\tau^n)Dist(T,I)$. By applying the limit we observe $d(i_n,\lambda_\sigma i_n)) \longrightarrow Dist(T,I)$.

Theorem 7. Suppose that convex metric space (Ω, d, ∇) and a map $\lambda: T \cup I \longrightarrow T \cup I$ has non empty subsets T and U of metric space Ω then it is considered as generalized cyclic enriched Bianchini contraction map, then there is any initial point $i_o \in T \cup I$, so we obtain $d(i_n,\lambda_\sigma i_n) \longrightarrow Dist(T,I)$, where $i_{n+1} = \nabla(i_n,\lambda i_n;\sigma) = \lambda_\sigma i_n$, $n \geq 0$. if $\{i_{2n}\}$ has convergent subsequences in T then there lies an element ω for this $d(\omega,\lambda_\sigma\omega) = Dist(T,I)$.

Proof:

Let obsereve if the sequence $\{i_{2n}\}$ has convergent subsequence $\{i_{2n(a)}\}$, as it is clear that Ω is generalized cyclic Bianchini enriched map, it entails that

$$d(i2n(a),\lambda\sigma\omega) = d(\lambda\sigma i2n(a-1),\lambda\sigma\omega) \leq \tau \max d(i_{2n(a-1)},\lambda_\sigma i_{2n(a-1)}),d(\omega,\lambda_\sigma\omega) + (1-\tau)\, Dist(T,I) \tag{9}$$

Now by the Equation (9), we observe

$d(\omega,\lambda\sigma\omega) \leq d(\omega,i2n(a)) + d(i2n(a),\lambda\sigma\omega)) \leq \tau \max d(i_{2n(a-1)},\lambda_\sigma i_{2n(a-1)}),d(\omega,\lambda_\sigma\omega) + (1-\tau)Dist(T,I)$.

$Dist(T,I) \leq d(\omega,\lambda_\sigma\omega) \leq \tau Dist(T,I) + (1-\tau)Dist(T,I) = Dist(T,I)$

If $\max d(i2n(a-1),\lambda\sigma i2n(a-1)),d(\omega,\lambda\sigma\omega) = d(\omega,\lambda\sigma\omega)$, then

$d(\omega,\lambda_\sigma\omega) \leq \frac{1}{1-t}d\big(\omega,i_{2n(a)}\big) + \frac{1-t}{1-t}Dist(T,I) = \frac{1}{1-t}d\big(\omega,i_{2n(a)}\big) + Dist(T,I)$.

By applying limit $n \longrightarrow \infty$ it is proved that $d(\omega,\lambda_\sigma\omega) = Dist(T,I)$.

Theorem 8. Suppose that convex metric space (Ω, d, ∇) and a map $\lambda: T \cup I \longrightarrow T \cup I$ has non empty subsets T and U of metric space Ω and T is closed then it is considered as generalized cyclic (γ, l) enriched Biachini contraction map, then there is any initial point $i_o \in T \cup I$, and $i_{n+1}\lambda_\sigma i_n = (1-\sigma)\lambda i_n + \sigma\lambda i_n = \lambda_\sigma i_n$ has $\sigma=\frac{1}{\gamma+1}$ and for each $\epsilon > 0$ there lies positive number $N_o s.t m \geq n > N_o$.

Proof:

Suppose that $\epsilon > 0$, by using the Theorem 3 there is an integer N_1 that state $||i_{2n} - i_{2n+1}|| < Dist(T,I) + \epsilon$ and λ is considered as generalized cyclic (γ,l) enriched Biachini contraction map, thus it states

$||i_{2m(a)} - i_{2n(a)+1}|| \leq ||i_{2m(a)} - i_{2m(a+2)}|| + ||i_{2m(a+2)} - i_{2n(a+3)}|| + ||i_{2n(a+3)} - i_{2n(a+1)}|| = ||i_{2m(a)} - i_{2m(a+2)}|| + ||i_{2n(a+3)} - i_{2n(a+1)}|| + ||\lambda_\sigma i_{2m(a+1)} - \lambda_\sigma i_{2m(a+2)}|| \leq ||i_{2m(a)} - i_{2m(a+2)}|| + ||i_{2n(a+3)} - i_{2n(a+1)}|| + \tau \max[||i_{2m(a+1)},\lambda_\sigma i_{2m(a+1)}||,||i_{2n(a+2)}, \lambda_\sigma i_{2n(a+2)}||]$.

if $\max[||i_{2m(a+1)},\lambda_\sigma i_{2m(a+1)}||,||i_{2n(a+2)},\lambda_\sigma i_{2n(a+2)}||] = ||i_{2m(a+1)},\lambda_\sigma i_{2m(a+1)}||$, then

$||i_{2m(a)}-i_{2n(a)+1}|| \leq ||i_{2m(a)}-i_{2m(a+2)}||+||i_{2n(a+3)}-i_{2n(a+1)}||+\tau \max[||i_{2m(a+1)},\lambda_\sigma i_{2m(a+1)}||+(1-\tau)Dist(T,I)$.

By applying the limit $a \longrightarrow \infty$, we obtain

$Dist(T,I) + \epsilon \leq \tau Dist(T,I) + (1-\tau)Dist(T,I)$.

This completes the proof.

Theorem 9. Suppose that convex metric space (Ω, d, ∇) and a map $\lambda: T \cup I \longrightarrow T \cup I$ has non empty subsets T and U of metric space Ω and T is closed then it is considered as generalized cyclic Binchini (γ,l) enriched contraction map, then there is best proximity point $\omega \in T$, in addition if $i_o \in T$, and $i_{n+1}\lambda_\sigma i_n = (1-\sigma)\lambda i_n + \sigma\lambda i_n = \lambda_\sigma i_n$ has $\sigma = \frac{1}{\gamma+1}$ then the sequence i_{2n} possess best proximity point ω.

Proof:

It is obvious that λ is generalized Binchini (γ,l) enriched contraction mapping. There lies $\gamma \in [0,\infty)$ with $l = [0,\frac{1}{3}]$, so that

$||\gamma(i-j) + \lambda i - \lambda j|| \leq \tau \max(||i-j|| + ||i-\lambda i|| + ||j-\lambda j||) + (1-\tau)Dist(T,I)$.

For the value $\gamma = 0$ here comes $l=1$, we obtain

$||\lambda i - \lambda j|| \leq \tau \max(||i-j|| + ||i-\lambda i|| + ||j-\lambda j||) + (1-\tau)Dist(T,I)$.

The following output are generated from [28].

For $\gamma > 0$ and $\sigma = \frac{1}{\gamma+1}$ then, we observe

$||\lambda_\sigma i - \lambda_\sigma j|| \leq \tau \max(\sigma||i-j|| + ||i-\lambda_\sigma i|| + ||j-\lambda_\sigma j||) + \sigma(1-\tau)Dist(T,I) < l(||i-j|| + ||i-\lambda_\sigma i|| + ||j-\lambda_\sigma j||) + (1-\tau)Dist(T,I)$.

Assume that $Dist(T,I) = 0$, then by utilizing Theorem 1, we get

$$\lim ||i_{2n} - \lambda_\sigma i_{2n}|| = Dist(T,I) \tag{10}$$

also there is some N_o and for every $m > n > N_o$,

$$||i_{2m} - \lambda_\sigma i_{2n}|| < Dist(T,I) + \epsilon \qquad (11)$$

By observing inequality Equations (10) and (11) we deduced that the sequence is cauchy and its converges thus it implies that it possess best proximity poiny.

Now it is being proved that best proximity point is unique and for this we take two different points p and q. Thus, $||q - \lambda_\sigma q|| = Dist(T,I)$ and $||p - \lambda_\sigma p|| = Dist(T,I)$. Now begin with hypotheses that result in conflicts.

$||p - \lambda_\sigma q|| \le ||p - i_{2n+1}|| + ||i_{2n+1} - \lambda_\sigma q|| < \tau \max||i_{2n} - \lambda_\sigma i_{2n}||,||q - \lambda_\sigma q|| + (1 - \tau)Dist(T,I)$.

If $\max||i_{2n} - \lambda_\sigma i_{2n}||,||q - \lambda_\sigma q|| + (1 - \tau)Dist(T,I) = ||i_{2n} - \lambda_\sigma i_{2n}||$, then

$||p - \lambda_\sigma q|| \le ||p - i_{2n+1}|| + \tau||i_{2n} - \lambda_\sigma i_{2n}|| + (1 - \tau)Dist(T,I)$.

As a tends to ∞, then

$Dist(T,I) \le ||p - \lambda_\sigma q|| < \tau Dist(T,I) + (1 - \tau)Dist(T,I) = Dist(T,I)$.

Again if $\max||i_{2n} - \lambda_\sigma i_{2n}||,||q - \lambda_\sigma q|| + (1 - \tau)Dist(T,I) = ||q - \lambda_\sigma q||$, then

$||p - \lambda_\sigma q|| \le ||p - i_{2n+1}|| + \tau||q - \lambda_\sigma q|| + (1 - \tau)Dist(T,I)$.

Hence, it states that p = q, so the uniqueness of the best proximity verified.

5. Application

The main results obtained above provide a solid theoretical foundation for practical implementations. In this section, we demonstrate how the developed framework can be applied to concrete problem, highlighting the applicability of cyclic enriched contraction mappings in relevant mathematical model. Consider the initial value problem.

$$v'(t) = H(t,v(t)),\ v(t_0) = v_0, \qquad (12)$$

where $H: I\times R \rightarrow R$ is a continuous function and t_0 is an interior point of the compact interval $I = [a,b]$.

Let C(I) denote the Banach space of all continuous real-valued functions on I, endowed with the supremum norm

$// v // = \sup|v(t)|$.

A function $v \in C(I)$ is a solution of Equation (12) if and only if it satisfies the integral equation,

$$v(t) = v_0 + \int_{t_0}^{t} H\big(s,v(s)\big)ds. \qquad (13)$$

Define an operator $T: C(I)\rightarrow C(I)$ by $(Tv)(t) = v_0 + \int_{t_0}^{t} H\big(s,v(s)\big)ds$.

Then fixed points of the operator T are precisely solutions of the initial value problem Equation (12).

Assume that there exists a constant $P_0 > 0$, such that

$$||H(tu_1)\text{-}H(tu_2)|| \le p_0\big|\big|v_1\text{-}v_2\big|\big|\ \forall t \in I \text{ and } u_1,u_2 \in R \qquad (14)$$

Under this condition, the operator T satisfies a cyclic enriched contraction-type condition on suitable subsets of C(I). Consequently, by applying the main fixed point theorem established in this paper, the operator T admits a unique fixed point in C(I).

Therefore, the initial value problem Equation (12). has a unique continuous solution on the interval I.

Lemma 6. Let u(t) is an outcome of an initial value problem

$'v\ (t) = H(i,v(t),v(t_0) = v_0$, if $(Tv)=v_0+\int_{t_0}^{t} H(s,v(n))dn$.

Assume the norm on C(I) and it is defined as $||m|| = \sup_{v\in I} m(t)$.

Theorem 10. Suppose that $H: I \times R \longrightarrow R$ is considered as continous function with t_o be an interior point of I.

Suppose there is some$P_o \ge 0$, then H fulfil the following:

$|H(v,i_1)-H(vi_2)| \le P_o|i_1 - i_2|$,

the variable $u \in R$ and $P_o \in I$.

It is clear that it posses the unique solution $u \in C(I)$.

Proof:

Define an operator T: C(I)→C(I) by $(Tv)(t) = v_0+\int_{t_0}^{t} H\big(s,v(s)\big)ds$.

Since H is continuous and v ∈ C(I), the function H(t,v(t)) is continuous on I. Hence, Tv ∈ C(I) and T is well defined.

$|Tv_1(t)-Tv_2(t)|=\left|\int_{t_0}^{t} H\big(s,v_1(s)\big)-H\big(s,v_2(s)\big)ds\right|\leq\int_{t_0}^{t}|H\big(s,v_1(s)\big)-H\big(s,v_2(s)\big)|ds\leq\int_{t_0}^{t} p_0|\big(v_1(s)\big)-\big(v_2(s)\big)|ds\leq p_0|t-t_0|\,||v_1-v_2||$.

Let v_1,v_2 ∈ C(I). For any t ∈ l we have

taking the supremum over t ∈ I, we obtain

$/\!/ Tv_1 - Tv_2 /\!/ \leq p_0(b-a) /\!/ v_1 - v_2 /\!/$.

Thus, the operator T satisfies a cyclic enriched contraction-type condition on appropriate subsets of C(I). By the main cyclic enriched contraction fixed point theorem established earlier in this paper, *T* admits a unique fixed point v ∈ C(I).

By Lemma 6, this fixed point is the unique solution of the integral Equation (13) and hence the unique solution of the initial value problem Equation (12).

6. Conclusion

This paper carried out the following new findings:

(1) In this article, we employed the successive strategy given by ishi in Example 3 to figure out specific points about generalized mappings.

(2) The iterative approach serves to demonstrate various convergence findings.

(3) A few computational experiments were conducted that reinforce the research’s primary assumptions.

(4) A numerical illustration shows that the ishi-iterative method for handling generalized contraction mappings has higher converging rates than some of the currently available techniques.

(5) The results obtained for cyclic enriched contraction mappings can be further extended beyond the current framework. In particular, the proposed iterative scheme may be applied to fractional differential equations to establish existence and convergence of solutions. Additionally, its applicability to variational inclusions and related equilibrium problems provides a promising direction for future research.

Conflicts of Interest

The authors declare no conflict of interest.

Generative AI Statement

The authors declare that no Generative AI was used in the creation of this manuscript.

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