Global Integrated Mathematics
https://gim.cultechpub.com/gim
Cultech Publishing

Article

Results Concerning to Existence for Impulsive Functional Integrodifferential Equation of First Order in Banach Spaces via Resolvent Operators

Kamalendra Kumar[1,*], Manish Nath Tripathi[2]

[1]Department of Basic Science, Shri Ram Murti Smarak College of Engineering and Technology, Bareilly, India

[2]Department of Mathematics, Madan Mohan Malviya Postgraduate College Bhatpar Rani, Deoria, India

[*]Corresponding author: Kamalendra Kumar, kamlendra.14kumar@gmail.com

Abstract

In this paper, we presented the existence of mild solutions for a class of first order non local impulsive functional integrodifferential equations with time varying delay (NIFIETVD) in Banach spaces. The investigation is executed within the framework of resolvent operator theory along with fixed point approaches, particularly the Leray-Schauder nonlinear alternative and Banach contraction principle. Sufficient criteria for guaranteeing the existence of solution are established by imposing suitable continuity, compactness and growth conditions on the related nonlinear operator. In contrast to many previous results, the analysis permits the nonlocal term to satisfy weaker continuity assumptions, hence widening the scope of applicability. Moreover, the analysis emphasizes how impulsive effects and nonlocal conditions can be steered within a unified framework, thus enriching the previous existence results for functional integrodifferential equations (FIE). At last, to validate the abstract results, a concrete application is stated, illustrating the usefulness of the derived outcomes.

Keywords

NIFIETVD, Resolvent operators, Impulsive conditions, Nonlocal conditions, Leray-Schauder theorem

MSC 2020: 34K45, 34G20, 34K30

Article History

Received: 29 September 2025
Revised: 14 November 2025
Accepted: 19 November 2025
Available Online: 08 December 2025

1. Introduction and Problem Statement

Due to the wide range of theoretical importance and practical applications in many realms of science and engineering, the investigation of functional and integrodifferential equations in Banach space have attracted very much attention of the researcher see for instance [1,2]. A lot of real world phenomena emerging in control theory, physics, population dynamics, biology, and economic are better explained by models that consist sudden changes and memory effects. To tackle these complexities investigators have turned to impulsive FIE that combine key characteristic such as integral terms to describe hereditary effects, delay effects and the impulsive conditions to reflect sudden and discontinuous changes [3-5].

In modern applied mathematics, impulsive effects are of particular significance. They model sudden changes in the state of a system, see for instance, switching phenomena in electrical circuits, dosing in pharmacokinetics, and harvesting in biological populations, notice the monograph of Samoilenko and Perestyuk [6], and in [7-11] and the references cited therein. Classical methods of ordinary differential equation that depend solely on continuous evolution are inappropriate to explain such types of behaviors. Thus, impulsive functional differential or integrodifferential equations (IE) have become a necessary part of nonlinear analysis obtaining increasing interest from both practitioners and theoreticians.

On the other hand, a class of IE is a significant class of functional differential equations which describe many dynamical systems containing nuclear reactor physics, visco-elastic fluids and population dynamics as discussed extensively in [12]. Additional examples include models of brain dynamics [13,14], as well as infectious disease spreading [15].

To investigate the equations of the type NIFIETVD, Banach spaces provides a unified and abstract framework which is appropriate for treating system governed by partial differential equation (PDE) with memory and impulses. Under this setting, the main challenge is to derive the conditions for the existence, uniqueness and other qualitative properties of solutions. Last many decades different types of techniques have been evolved to address these problems. These consist of the use of fixed point approaches such as Banach, Schaefer's, Schauder and Krasnoselskii's fixed-point theorem [16,17], the methods based on semigroup theory [18] and the use of measure of noncompactness [19,20]. However, the resolvent operator approach is applied in relatively few works, especially for first order NIFIETVD in Banach spaces. Within weaker assumptions this operator-theoretic approach generalizes the previous results as well as offer more refined tools for deriving existence theorems.

In contrast to initial conditions, that characterize the state of a system at a single point, nonlocal conditions take part in the solution over an interval or through integral relations. These conditions generally comes in to picture where the present state depends on past effects, as in population dynamics, control theory and stability analysis. It give a more realistic framework than the classical conditions. Due to the above advantage over the classical conditions, nonlocal condition extensively investigated for numerous class of differential and IE, leading to important advances in existence and uniqueness theory. Byszewski started the investigation of nonlocal initial value problems. He used the semigroup approach to derive the existence, uniqueness of mild, classical and strong solution of various kinds of abstract differential [21,22]. Since then, many researchers [23-25] investigated the abstract nonlocal conditions together with differential and integrodifferential equations in abstract spaces.

Time varying delay signifies a situation in dynamic system in which the delay in the system's response changes dynamically rather than remaining constant. These delays based on factors like current state of the system, environmental conditions or external inputs, making the system behavior more complex and challenging to anticipate. Such delays generally present in real phenomena, including communication network, control systems and biological systems, where transmission, processing or reaction times fluctuate. In mathematical models, time varying delay are represented as system variable or function of time that significantly impact the stability, controllability and the existence of solutions. Carefully taken in to account of these delays is important, as even small fluctuations may be the lead to instability or unexpected dynamics, requiring developed analytical and computational approaches for effective management. Many authors explored the existence of solution for differential or integrodifferential equation with time varying delays of various kind, see [26-28].

Several authors [29,30] established the existence results for various class of impulsive integrodifferential equation with nonlocal conditions assuming both delayed and non-delayed cases. Kumar et al. [31] have investigated nonlinear IE along with time dependent delay in Banach spaces. It derived the conditions for the existence and uniqueness by employing semigroup approach. Chadha and Pandey [32] studied a class of non-autonomous functional differential equation of neutral type associated with impulsive and nonlocal condition in Banach space. It derive the existence of mild solutions using approximation techniques combined with Krasnoselskii's fixed point theorem. The method overcome strict conditions on impulsive and nonlocal term. In [33], author studied the nonlinear IE with time varying delay in Banach spaces. Applying Schaefer fixed point theorem in association with resolvent operator author established conditions assuring the existence of solutions. In this manuscript we have investigated nonlocal Cauchy problems for nonlinear IE with resolvent operators offers an abstract formulation of partial integrodifferential equation that emerged in several applications like heat equations, viscoelasticity, and several other physical models [23,34,35].

Inspired by the work that is presented above, we suppose that NIIETVD of the following form:

$$\omega'(t) = A(t)\left[\omega(t) + \int_0^t \Upsilon(t,\hbar)\omega(\hbar)d\hbar\right]$$
$$+P_1\left(t, \omega(d_1(t)),...,\omega(d_n(t)), \int_0^t J_1\left(t,\hbar,\omega\left(d_{n+1}(\hbar)\right)\right)d\hbar\right) \quad (1)$$
$$+P_2\left(t, \omega(\varphi_1(t)),...,\omega(\varphi_m(t)), \int_0^t J_2\left(t,\hbar,\omega\left(\varphi_{m+1}(\hbar)\right)\right)d\hbar\right)$$

$t \in \Xi = [0,c], t \neq t_i, i = 1,2,...,k$,

$$\omega(0) + \varsigma(\omega) = \omega_0, \quad (2)$$
$$\Delta\omega(t_i) = \wp_i\left(\omega(t_i)\right), i = 1,2,...,k, \quad (3)$$

where the unknown function $\omega(.)$ which takes its values in V, a Banach space. Here the operator $A(t)$ is supposed to be closed and linear in V, whose domain $D(A)$ is dense in V that remains fixed for all t. Consider $0 = t_0 < t_1 < ... < t_k < t_{k+1} = c$ be a finite sequence of prescribed points. At each point $t = t_i\,(i = 1,2,...,k)$, the discontinuities of $\omega(t)$ is represented as $\Delta\omega(t_i) = \omega(t_i^+) - \omega(t_i^-)$, where the notations $\omega(t_i^+), \omega(t_i^-)$ mark the right and left limits of $\omega(t)$. The nonlinear operator are given by $P_1 : \Xi \times V^{n+1} \to V, P_2 : \Xi \times V^{m+1} \to V$, $J_1, J_2 : \Xi \times \Xi \times V \to V$, $\varsigma : PC(\Xi, V) \to V$, along with given functions $d_l : \Xi \to \Xi, l = 1,2,...,n+1$ and $\varphi_p : \Xi \to \Xi,\ p = 1,\ 2,...,m+1$. The linear operator $\Upsilon(t,\hbar)$ which is bounded in V for each $t,\hbar \in \Xi$.

The presented paper is devoted to meet our principal goal in which we have investigated the NIFIETVD problem described in (1)–(3) within the structure of Banach spaces. Besides of the following conventional route, the study utilizes an association of advanced analytical tools–namely, the Banach contraction principle, the Leray–Schauder nonlinear alternative, and the structure of resolvent operators–to validate the existence of mild solutions.

2. Preliminaries and Fundamental Concepts

In the sequence of derive our main results, we begin by presenting some auxiliary results, notations, and basic definitions in which our analysis in the preceding section will take place.

Consider $C(\Xi, V)$ represent the Banach space that consisting of each continuous mappings $\omega : \Xi \to V$ under the supremum norm

$$\|\omega\|_\infty = sup\left\{\|\omega(t)\| : t \in \Xi, \omega \in C(\Xi, V)\right\}.$$

The collection of bounded linear operators from V to V is expressed by $B(V)$ with the norm

$$\|U\|_{B(V)} = \sup\left\{\|U(\omega)\| : \|\omega\| = 1\right\}.$$

A function $\omega : \Xi \to V$ is a Bochner integrable rigorously if its norm $\|\omega\|$ is Lebesgue integrable (for a rigorous treatment of this concept, one may refer to [36]). Moreover, we denote by $L^1(\Xi, V)$ the Banach space consisting of all measurable mappings $\omega : \Xi \to V$ that are Bochner integrable. The space is equipped with the norm

$$\|\omega\|_{L^1} = \int_0^e \|\omega(t)\| dt, \omega \in L^1(\Xi, V).$$

To study the family of linear operators $\{A(t) : 0 \le t \le c\}$, we need the following conditions (see [37]).

(a) Resolvent conditions: $\forall t \in \Xi$, the resolvent $R(\lambda, A(t))$ exists $\forall \lambda$ such that $\mathrm{Re}\lambda \ge 0$. $\exists$ a constant $N^* > 0$ in such a way that

$$\left\|R(\lambda, A(t))\right\| \le \frac{N^*}{(|\lambda| + 1)}.$$

(b) Bound on operator difference: For any $t, \hbar, \zeta \in \Xi$, there exists a constant $0 < \chi < 1$ and $N^* > 0$ in such a way that

$\left\|\left(A(t)-A(\zeta)\right)A^{-1}(\hbar)\right\|\leq N^{*}|t-\zeta|^{\chi}$.

Moreover, $\forall t\in\Xi$ and $\exists\lambda\in\rho A(t)$, the resolvent set $R(\lambda,A(t))$ is compact.

(c) Domain and operator properties: Suppose that $D(A)$ represents the domain of the family of operators $\{A(t):0\leq t\leq c\}$. Consider that $D(A)$ is dense in V and does not depend on t, $A(t)$ is a closed linear operator.

In sequence to explain the solution structure of the problem (1)-(3), let us assume the function space $PC([0,c],V)=\{\omega:\Xi\to V:\omega(t)$ is continuous for $t\neq t_i$, left continuous at $t=t_i$, and has the right hand limit $\omega(t_i^{+})$ for $i=1,2,...,k\}$. This is a Banach space equipped with norm

$$\|\omega\|_{PC}=\sup_{\varsigma\in\Xi}\|\omega(t)\|.$$

For convenience the notations, we denote $t_0=0$, $t_{m+1}=c$. $\forall\omega\in PC([0,c],V)$ we represent by

$$\tilde{\omega}_i(t)=\begin{cases}\omega(t),t\in(t_i,t_{i+1}]\\ \omega(t_i^{+}),t=t_i\end{cases}.$$

Also, for $E\subseteq PC([0,c],V)$ we represent by $\tilde{E}_i, i=0,1,...,k$, the set $\tilde{E}_i=\{\tilde{\omega}_i:\omega\in E\}$.

To prepare the essential foundation of our primary result, we now state the following lemmas and definitions.

Lemma 1: "Suppose a subset E of $PC([0,c],V)$. The set E is relatively compact in $PC([0,c],V)$ precisely when, $\forall i=1,2,...,k$, the associated set $\tilde{E}_i$ is relatively compact in $C([t_i,t_{i+1}];V)$."

Lemma 2 [38]**:** "If the condition (i)-(iii) fulfilled, then the operator family $\{A(t),t\in[0,c]\}$ produces a uniquely determined linear evolution system $\{F(t,\hbar):0\leq\hbar\leq t\leq c\}$. Furthermore, if $t>\hbar$ together $0\leq\hbar<t\leq c$, then the mapping $F(t,\hbar)$ is a compact linear operator on V."

Lemma 3 [39]**:** "Let V be a Banach space and consider $\Omega\subset V$ be a convex set which is closed. Consider a relatively open subset $S\subset\Omega$ with $0\in S$. If $H:\overline{S}\to\Omega$ is a compact mapping, then one of the following hold:

(a) $\exists u\in\partial S$ and a constant $\iota\in(0,1)$ such that $u=\iota H(u)$

(b) H has a fixed point in $\overline{S}$."

Definition 1 [40-42]**:** A resolvent operator associated to NIIETVD (1)-(3) is defined as a bounded operator valued function $F(t,\hbar)\in B(V), 0\leq\hbar\leq t\leq c$, where $B(V)$ expressed as the space of bounded linear operators on V. The operator $F(t,\hbar)$ meets the following conditions:

(a) $F(t,\hbar)$ is strongly continuous in both the variables $\hbar$ and t, $F(\hbar,\hbar)=I$, $0\leq\hbar\leq c$, $\|F(t,\hbar)\|\leq C_1e^{\gamma(t-\hbar)}$ for some constants C_1 and γ.

(b) The operator $F(t,\hbar)$ maps Z into itself that is $F(t,\hbar)Z\subset Z$, and $F(t,\hbar)$ is strongly continuous in $\hbar$ and t on Banach space Z. Here Z represents the Banach space that is formed from the domain $D(A)$ of $A(t)$, equipped with the graph norm.

(c) $\forall\omega\in V$, the mapping $F(t,\hbar)$ is continuously differentiable in both $t,\hbar\in\Xi$ and the derivative is stated by

$$\frac{\partial F}{\partial t}(t,\hbar)\omega=A(t)\left[F(t,\hbar)\omega+\int_{\hbar}^{t}\Upsilon(t,\Im)F(\Im,\hbar)\omega d\Im\right]$$

Definition 2: A function $\omega(.):\Xi\to V$ is called a mild solution of (1)-(3) if it is continuous and $\forall\omega_0\in V$, it fulfills the associated integral equation

$$\omega(t)=F(t,0)\left[\omega_0-\varsigma(\omega)\right]+\sum_{0<t_i<t}F(t,t_i)\wp_i\left(\omega(t_i)\right)$$
$$+\int_0^t F(t,\hbar)\left[P_1\left(\hbar,\omega\left(d_1(\hbar)\right),...,\omega\left(d_n(\hbar)\right),\int_0^\hbar J_1\left(\hbar,\Im,\omega\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right]d\hbar \tag{4}$$
$$+\int_0^t F(t,\hbar)\left[P_2\left(\hbar,\omega\left(\varphi_1(\hbar)\right),...,\omega\left(\varphi_m(\hbar)\right),\int_0^\hbar J_2\left(\hbar,\Im,\omega\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right]d\hbar .$$

In addition, the following assumptions are considered:

(A_1) $\forall t,\hbar>0$, the resolvent operator $F(t,\hbar)$ is compact.

(A_2) The functions $P_1:\Xi\times V^{n+1}\to V$ and $P_2:\Xi\times V^{m+1}\to V$ are continuous. $\exists$ constants $\widehat{C}_2,\widehat{C}_3>0,\breve{C}_2,\breve{C}_3\geq 0$ in such a way that $\forall \bar{v}_l,\bar{w}_l\in V, l=1,2,...,n+1$ and $\bar{v}_p,\bar{w}_p\in V, p=1,2,...,m+1$, we have

$$\left\|P_1\left(t,\bar{v}_1,\bar{v}_2...,\bar{v}_{n+1}\right)-P_1\left(t,\bar{w}_1,\bar{w}_2...,\bar{w}_{n+1}\right)\right\|\leq \widehat{C}_2\left(\sum_{l=1}^{n+1}\left\|\bar{v}_l-\bar{w}_l\right\|\right),$$

$$\left\|P_2\left(t,\bar{v}_1,\bar{v}_2...,\bar{v}_{n+1}\right)-P_2\left(t,\bar{w}_1,\bar{w}_2...,\bar{w}_{m+1}\right)\right\|\leq \widehat{C}_3\left(\sum_{p=1}^{m+1}\left\|\bar{v}_p-\bar{w}_p\right\|\right),$$

and

$$\breve{C}_2=\max_{t\in\Xi}\left\|P_1(t,0,...,0)\right\|,$$

$$\breve{C}_3=\max_{t\in\Xi}\left\|P_2(t,0,...,0)\right\|.$$

(A_3) The functions $J_1,J_2:\Xi\times\Xi\times V\to V$ are continuous. $\exists$ constants $C_4,C_5>0,\ \ \breve{C}_4,\breve{C}_5\geq 0$ in such a way that $\forall\bar{v},\bar{w}\in V$,

$$\left\|J_1\left(t,\hbar,\bar{v}\right)-J_1\left(t,\hbar,\bar{w}\right)\right\|\leq C_4\left\|\bar{v}-\bar{w}\right\|,$$

$$\left\|J_2\left(t,\hbar,\bar{v}\right)-J_2\left(t,\hbar,\bar{w}\right)\right\|\leq C_5\left\|\bar{v}-\bar{w}\right\|,$$

and

$$\breve{C}_4=\max_{0\leq\hbar\leq t\leq e}\left\|J_1(t,\hbar,0)\right\|,$$

$$\breve{C}_5=\max_{0\leq\hbar\leq t\leq e}\left\|J_2(t,\hbar,0)\right\|.$$

(A_4) The functions $d_l:\Xi\to\Xi, l=1,2,...,n+1$ and $\varphi_p:\Xi\to\Xi, p=1,2,...,m+1$ are continuous and fulfill the conditions $d_l(t)\leq t, l=1,2,...,n+1$ and $\varphi_p(t)\leq t, p=1,2,...,m+1$.

(A_5) $\wp_i\in C(V,V), i=1,2,...,k$ are all compact operators. It consists continuous nondecreasing functions $\rho_i:[0,\infty)\to(0,\infty)$, $i=1,2,...,k$, in such a way that

$$\left\|\wp_i(v)\right\|\leq\rho_i\left(\|v\|\right),\forall v\in V.$$

(A_5) The function $\varsigma(.):PC(\Xi,V)\to V$ is considered to be continuous. $\exists$ a number $\varepsilon\in(0,c)$ in such a way that $\varsigma(\sigma)=\varsigma(\tilde{\sigma})$, $\forall\sigma,\tilde{\sigma}\in PC(\Xi,V)$, if $\sigma(t)=\tilde{\sigma}(t),\forall t\in(\varepsilon,c)$. Furthermore, there exists continuous nondecreasing function $\Lambda:[0,\infty)\to(0,\infty)$ in such a way that

$$\left\|\varsigma(\sigma)\right\|\leq\Lambda\left(\|\sigma\|_{PC}\right),\sigma\in PC(\Xi,V).$$

(A_6) A positive constant $Q^*>0$ in such a way that

$$\frac{Q^*}{C_1\left[\|\omega_0\|+\Lambda Q^*+\sum_{i=1}^{k}\rho_i Q^*+\frac{\left\{c\left(\widehat{C}_2\breve{C}_4+\widehat{C}_3\breve{C}_5\right)+\left(\breve{C}_2+\breve{C}_3\right)\right\}}{\gamma}\right]D_0}>1$$

where $D_0=e^{\left\{C_1\left[\left(\widehat{C}_2 n+\widehat{C}_3 m\right)+c\left(\widehat{C}_2 C_4+\widehat{C}_3 C_5\right)\right]+\gamma\right\}c}$.

3. Main Results

Theorem 1: Let us assume that $\omega(0)\in V$. Again, consider that assumptions $(A_1)-(A_6)$ met. Then, the NIIETVD problem stated by (1)-(3) admits at least one mild solution on the interval Ξ.

Proof: Let us initiate by setting $C_0 := C_1\left[\left(n\widehat{C}_2+\widehat{C}_3 m\right)+\left(\widehat{C}_2 C_4+\widehat{C}_3 C_5\right)c\right]+\gamma$. In order to proceed, we set in the space $PC(\Xi,V)$ the equivalent norm described as

$$\|\sigma\|_Y := \sup_{t\in\Xi} e^{-C_0 t}\|\sigma(t)\|.$$

With this choice, one can readily confirm that $Y:=\left(PC(\Xi,V),\|.\|_Y\right)$ is any Banach space. Now, let us fix $\Theta\in PC(\Xi,V)$. $\forall t\in\Xi,\sigma\in Y$, we now proceed to introduce an operator

$$\begin{aligned}(W_\Theta\sigma)(t)=&F(t,0)\left[\omega_0-\varsigma(\Theta)\right]+\sum_{0<t_i<t}F(t,t_i)\wp_i\left(\Theta(t_i)\right)+\int_0^t F(t,\hbar)\\ &\times\left[P_1\left(\hbar,\sigma\left(d_1(\hbar)\right),\dots,\sigma\left(d_n(\hbar)\right),\int_0^\hbar J_1\left(\hbar,\Im,\sigma\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right. \\ &\left.+P_2\left(\hbar,\sigma\left(\varphi_1(\hbar)\right),\dots,\sigma\left(\varphi_m(\hbar)\right),\int_0^\hbar J_2\left(\hbar,\Im,\sigma\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right]d\hbar.\end{aligned}\tag{5}$$

Noting that $F(.,0)\left(\omega_0-\varsigma(\Theta)\right)\in PC(\Xi,V)$, it follows, from condition $(A_1)-(A_4)$ that $(W_\Theta\sigma)(t)\in Y,\forall\sigma\in Y$. Consider $\sigma,\tilde{\sigma}\in Y$. We have

$$\begin{aligned}&e^{-C_0 t}\left\|(W_\Theta\sigma)(t)-(W_\Theta\tilde{\sigma})(t)\right\|\\ &\le e^{-C_0 t}\int_0^t\left\|F(t,\hbar)\left[P_1\left(\hbar,\sigma\left(d_1(\hbar)\right),\dots,\sigma\left(d_n(\hbar)\right),\int_0^\hbar J_1\left(\hbar,\Im,\sigma\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right.\right.\\ &\left.\left.-P_1\left(\hbar,\tilde{\sigma}\left(d_1(\hbar)\right),\dots,\tilde{\sigma}\left(d_n(\hbar)\right),\int_0^\hbar J_1\left(\hbar,\Im,\tilde{\sigma}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right]\right\|d\hbar\\ &+e^{-C_0 t}\int_0^t\left\|F(t,\hbar)\left[P_2\left(\hbar,\sigma\left(\varphi_1(\hbar)\right),\dots,\sigma\left(\varphi_m(\hbar)\right),\int_0^\hbar J_1\left(\hbar,\Im,\sigma\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right.\right.\\ &\left.\left.-P_2\left(\hbar,\tilde{\sigma}\left(\varphi_1(\hbar)\right),\dots,\tilde{\sigma}\left(\varphi_m(\hbar)\right),\int_0^\hbar J_2\left(\hbar,\Im,\tilde{\sigma}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right]\right\|d\hbar\\ &\le C_1\widehat{C}_2\int_0^t e^{-C_0 t}e^{\gamma(t-\hbar)}\left[\left\|\sigma\left(d_1(\hbar)\right)-\tilde{\sigma}\left(d_1(\hbar)\right)\right\|+\dots+\left\|\sigma\left(d_n(\hbar)\right)-\tilde{\sigma}\left(d_n(\hbar)\right)\right\|\right.\\ &\left.+\left\|\int_0^\hbar J_1\left(\hbar,\Im,\sigma\left(d_{n+1}(\Im)\right)\right)d\Im-\int_0^\hbar J_1\left(\hbar,\Im,\tilde{\sigma}\left(d_{n+1}(\Im)\right)\right)d\Im\right\|\right]d\hbar\\ &+C_1\widehat{C}_3\int_0^t e^{-C_0 t}e^{\gamma(t-\hbar)}\left[\left\|\sigma\left(\varphi_1(\hbar)\right)-\tilde{\sigma}\left(\varphi_1(\hbar)\right)\right\|+\dots+\left\|\sigma\left(\varphi_m(\hbar)\right)-\tilde{\sigma}\left(\varphi_m(\hbar)\right)\right\|\right.\\ &\left.+\left\|\int_0^\hbar J_2\left(\hbar,\Im,\sigma\left(\varphi_{m+1}(\Im)\right)\right)d\Im-\int_0^\hbar J_2\left(\hbar,\Im,\tilde{\sigma}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right\|\right]d\hbar\\ &\le C_1\widehat{C}_2\int_0^t e^{-C_0 t}e^{\gamma(t-\hbar)}\left[\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|+\dots+\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|\right.\end{aligned}$$

$$+C_4\int_0^{\hbar}\left\|\sigma\left(d_{n+1}(\Im)\right)-\tilde{\sigma}\left(d_{n+1}(\Im)\right)\right\|d\Im\Big]d\hbar$$

$$+C_1\widehat{C}_3\int_0^t e^{-C_0t}e^{\gamma(t-\hbar)}\Big[\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|+...+\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|+C_5\int_0^{\hbar}\left\|\sigma\left(\varphi_{m+1}(\Im)\right)-\tilde{\sigma}\left(\varphi_{m+1}(\Im)\right)\right\|d\Im\Big]d\hbar$$

$$\le C_1\widehat{C}_2\int_0^t e^{-C_0t}e^{\gamma(t-\hbar)}\Big[n\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|+C_4c\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|\Big]d\hbar$$

$$+C_1\widehat{C}_3\int_0^t e^{-C_0t}e^{\gamma(t-\hbar)}\Big[m\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|+C_5c\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|\Big]d\hbar$$

$$\le C_1\widehat{C}_2\int_0^t e^{(C_0-\gamma)(\hbar-t)}\Big[(n+C_4c)\sup_{\hbar\in\Xi}e^{-C_0\hbar}\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|d\hbar$$

$$+C_1\widehat{C}_3\int_0^t e^{(C_0-\gamma)(\hbar-t)}\Big[(m+C_5c)\sup_{\hbar\in\Xi}e^{-C_0\hbar}\left\|\sigma(\hbar)-\tilde{\sigma}(\hbar)\right\|d\hbar$$

$$\le C_1\widehat{C}_2(n+C_4c)\int_0^t e^{(C_0-\gamma)(\hbar-t)}d\hbar\left\|\sigma-\tilde{\sigma}\right\|_Y+C_1\widehat{C}_3(m+C_5c)\int_0^t e^{(C_0-\gamma)(\hbar-t)}\left\|\sigma-\tilde{\sigma}\right\|_Y d\hbar$$

$$\le C_1\Big[\left(n\widehat{C}_2+\widehat{C}_3m\right)+\left(\widehat{C}_2C_4+\widehat{C}_3C_5\right)c\Big]\int_0^t e^{(C_0-\gamma)(\hbar-t)}d\hbar\left\|\sigma-\tilde{\sigma}\right\|_Y$$

$$\le\frac{C_1\Big[\left(n\widehat{C}_2+\widehat{C}_3m\right)+\left(\widehat{C}_2C_4+\widehat{C}_3C_5\right)c\Big]}{C_0-\gamma}\left\|\sigma-\tilde{\sigma}\right\|_Y,\ t\in\Xi,$$

which demonstrate that

$$e^{-C_0t}\left\|(W_\Theta\sigma)(t)-(W_\Theta\tilde{\sigma})(t)\right\|\le\frac{1}{2}\left\|\sigma-\tilde{\sigma}\right\|_Y,\ t\in\Xi.$$

Thus,

$$\left\|(W_\Theta\sigma)(t)-(W_\Theta\tilde{\sigma})(t)\right\|\le\frac{1}{2}\left\|\sigma-\tilde{\sigma}\right\|_Y,\ \sigma,\tilde{\sigma}\in Y.$$

This means that, W_Θ is a strict contractive, the Banach contraction principle employs, assuring that W_Θ possess a unique fixed point $\sigma_\Theta\in Y$. This, in turn, derives that the equation (5) contains a unique mild solution on $[0,c]$. Consider

$$\tilde{\Theta}(t)=\begin{cases}\Theta(t), \varepsilon<t\le c,\\ \Theta(\varepsilon), 0\le t\le\varepsilon.\end{cases}$$

From equation (5), we have

$$\sigma_{\tilde{\Theta}}(t)=F(t,0)\left[\omega_0-\varsigma\left(\tilde{\Theta}\right)\right]+\sum_{0<t_i<t}F(t,t_i)\wp_i\left(\Theta(t_i)\right)+\int_0^t F(t,\hbar)$$
$$\times\Bigg[P_1\left(\hbar,\sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right),\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)$$
$$+P_2\left(\hbar,\sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right),\int_0^{\hbar}J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\Bigg]d\hbar. \tag{6}$$

We introduce the map $I:PC_\varepsilon=PC\left([\varepsilon,c],V\right)\to PC_\varepsilon$ stated by

$$(I\Theta)(t)=\sigma_{\tilde{\Theta}}(t),\ \varepsilon\le t\le c. \tag{7}$$

Our aim is to establish that I holds every conditions of Lemma 3. The proof will be organized in successive steps.

Step 1. I maps bounded sets in PC_ε into bounded sets.

Choose $ƛ>0$ and define

$$\Theta\in C_{ƛ}(\varepsilon):=\left\{\sigma\in PC\left([\varepsilon,c],V\right);\sup_{\varepsilon\le t\le c}\left\|\sigma(t)\right\|\le ƛ\right\}.$$

It is suffice to demonstrate that $\exists\ M>0$ such that $\forall\Theta\in C_{ƛ}(\varepsilon)$ one has $\left\|I\Theta\right\|_{PC}\le M$.

Fix $\Theta \in C_{\lambda}(\varepsilon)$. Then, $\forall 0 < t \le c$, we have

$$e^{-\gamma t}\left\|\sigma_{\tilde{\Theta}}(t)\right\| \le e^{-\gamma t}\left\|F(t,0)\left[\omega_0 - \varsigma(\tilde{\Theta})\right]\right\| + \left\|\sum_{0<t_i<t} F(t,t_i)\wp_i\left(\Theta(t_i)\right)\right\|$$

$$+e^{-\gamma t}\int_0^t\left\|F(t,\hbar)P_1\left(\hbar,\sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right),\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right\|d\hbar$$

$$+e^{-\gamma t}\int_0^t\left\|F(t,\hbar)P_2\left(\hbar,\sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right),\int_0^{\hbar}J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right\|d\hbar$$

$$\le C_1\left[\left\|\omega_0 + \varsigma(\tilde{\Theta})\right\|\right] + C_1\sum_{i=1}^{k}\left\|\wp_i\left(\Theta(t_i)\right)\right\|$$

$$+C_1\int_0^t e^{-\gamma\hbar}\left[\left\|P_1\left(\hbar,\sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right),\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right.\right.$$

$$\left.\left.-P_1(\hbar,0,...,0)\right\| + \left\|P_1(\hbar,0,...,0)\right\|\right]d\hbar + C_1\int_0^t e^{-\gamma\hbar}\times$$

$$\left[\left\|P_2\left(\hbar,\sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right),\int_0^{\hbar}J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right) - P_2(\hbar,0,...,0)\right\|\right.$$

$$\left.+\left\|P_2(\hbar,0,...,0)\right\|\right.$$

$$\le C_1\left[\left\|\omega_0\right\| + \left\|\varsigma(\tilde{\Theta})\right\|\right] + C_1\sum_{i=1}^{k}\rho_i\left\|\left(\Theta(t_i)\right)\right\| + C_1\int_0^t e^{-\gamma\hbar}\left[\widehat{C}_2\left\{\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\| + ... + \left\|\sigma_{\tilde{\Theta}}(\hbar)\right\|\right.\right.$$

$$\left.\left.+\int_0^{\hbar}\left(\left\|J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right) - J_1(\hbar,\Im,0)\right\| + \left\|J_1(\hbar,\Im,0)\right\|\right)d\Im\right\} + \breve{C}_2\right]d\hbar$$

$$+C_1\int_0^t e^{-\gamma\hbar}\left[\widehat{C}_3\left\{\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\| + ... + \left\|\sigma_{\tilde{\Theta}}(\hbar)\right\|\right.\right.$$

$$\left.\left.+\int_0^{\hbar}\left(\left\|J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right) - J_2(\hbar,\Im,0)\right\| + \left\|J_2(\hbar,\Im,0)\right\|\right)d\Im\right\} + \breve{C}_3\right]d\hbar$$

$$\le C_1\left[\left\|\omega_0\right\| + \Lambda\left\|\tilde{\Theta}\right\|_{PC}\right] + C_1\sum_{i=1}^{k}\rho_i\left\|\left(\Theta(t_i)\right)\right\| + C_1\int_0^t e^{-\gamma\hbar}\left[\widehat{C}_2\left\{n\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\|\right.\right.$$

$$\left.\left.+c\left(C_4\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\| + \breve{C}_4\right)\right\} + \breve{C}_2\right]d\hbar + C_1\int_0^t e^{-\gamma\hbar}\left[\widehat{C}_3\left\{m\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\| + c\left(C_5\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\| + \breve{C}_5\right)\right\} + \breve{C}_3\right]d\hbar$$

$$\le C_1\left[\left\|\omega_0\right\| + \Lambda(t)\right] + C_1\sum_{i=1}^{k}\rho_i(t) + \frac{C_1\left(c\widehat{C}_2\breve{C}_4 + \breve{C}_2\right)}{\gamma} + C_1\left(\widehat{C}_2 n + \widehat{C}_2 cC_4\right)\int_0^t e^{-\gamma\hbar}\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\|d\hbar$$

$$+\frac{C_1\left(c\widehat{C}_3\breve{C}_5 + \breve{C}_3\right)}{\gamma} + C_1\left(\widehat{C}_3 m + \widehat{C}_3 cC_5\right)\int_0^t e^{-\gamma\hbar}\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\|d\hbar.$$

$$\le C_1\left[\left\|\omega_0\right\| + \Lambda(t)\right] + C_1\sum_{i=1}^{k}\rho_i(t) + \frac{C_1\left[c\left(\widehat{C}_2\breve{C}_4 + \widehat{C}_3\breve{C}_5\right) + \left(\breve{C}_2 + \breve{C}_3\right)\right]}{\gamma}$$

$$+C_1\left[\left(\widehat{C}_2 n + \widehat{C}_3 m\right) + c\left(\widehat{C}_2 C_4 + \widehat{C}_3 C_5\right)\right]\int_0^t e^{-\gamma\hbar}\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\|d\hbar$$

By applying the Gronwall's inequality

$$e^{-\gamma t}\left\|\sigma_{\tilde{\Theta}}(t)\right\| \le \left[C_1\left(\left\|\omega_0\right\| + \Lambda(t)\right) + C_1\sum_{i=1}^{k}\rho_i(t) + \frac{C_1\left\{c\left(\widehat{C}_2\breve{C}_4 + \widehat{C}_3\breve{C}_5\right) + \left(\breve{C}_2 + \breve{C}_3\right)\right\}}{\gamma}\right] \times e^{C_1\left[\left(\widehat{C}_2 n + \widehat{C}_3 m\right) + c\left(\widehat{C}_2 C_4 + \widehat{C}_3 C_5\right)\right]c}.$$

Consequently

$$\left\|I\Theta\right\|_{PC} \le \left[C_1\left(\left\|\omega_0\right\| + \Lambda(t)\right) + C_1\sum_{i=1}^{k}\rho_i(t) + \frac{C_1\left\{c\left(\widehat{C}_2\breve{C}_4 + \widehat{C}_3\breve{C}_5\right) + \left(\breve{C}_2 + \breve{C}_3\right)\right\}}{\gamma}\right] \times e^{\left\{C_1\left[\left(\widehat{C}_2 n + \widehat{C}_3 m\right) + c\left(\widehat{C}_2 C_4 + \widehat{C}_3 C_5\right)\right] + \gamma\right\}c} = M.$$

Step 2. $I: PC_\varepsilon \to PC_\varepsilon$ is continuous.

From (6) and assumptions $(A_1)-(A_5)$, we suppose that for $\Theta_1, \Theta_2 \in C_{\lambda}(\varepsilon)$, $0 < t \le c$,

$$e^{-\gamma t}\left\|\sigma_{\tilde{\Theta}_1}(t)-\sigma_{\tilde{\Theta}_2}(t)\right\| \le e^{-\gamma t}\left\|F(t,0)\left\{\varsigma(\tilde{\Theta}_1)-\varsigma(\tilde{\Theta}_2)\right\}\right\|$$

$$+\left\|\sum_{0<t_i<t}F(t,t_i)\wp_i(\Theta_1(t_i))-\sum_{0<t_i<t}F(t,t_i)\wp_i(\Theta_2(t_i))\right\|$$

$$+e^{-\gamma t}\int_0^t\left\|F(t,\hbar)\left[P_1\left(\hbar,\sigma_{\tilde{\Theta}_1}(d_1(\hbar)),\ldots,\sigma_{\tilde{\Theta}_1}(d_n(\hbar)),\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}_1}(d_{n+1}(\Im))\right)d\Im\right)\right.\right.$$

$$\left.\left.-P_1\left(\hbar,\sigma_{\tilde{\Theta}_2}(d_1(\hbar)),\ldots,\sigma_{\tilde{\Theta}_2}(d_n(\hbar)),\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}_2}(d_{n+1}(\Im))\right)d\Im\right)\right]\right\|d\hbar$$

$$+e^{-\gamma t}\int_0^t\left\|F(t,\hbar)\left[P_2\left(\hbar,\sigma_{\tilde{\Theta}_1}(\varphi_1(\hbar)),\ldots,\sigma_{\tilde{\Theta}_1}(\varphi_m(\hbar)),\int_0^{\hbar}J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}_1}(\varphi_{m+1}(\Im))\right)d\Im\right)\right.\right.$$

$$\left.\left.-P_2\left(\hbar,\sigma_{\tilde{\Theta}_2}(\varphi_1(\hbar)),\ldots,\sigma_{\tilde{\Theta}_2}(\varphi_m(\hbar)),\int_0^{\hbar}J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}_2}(\varphi_{m+1}(\Im))\right)d\Im\right)\right]\right\|d\hbar$$

$$\le C_1\left\|\varsigma(\tilde{\Theta}_1)-\varsigma(\tilde{\Theta}_2)\right\|+C_1\sum_{i=1}^{k}\left\|\wp_i(\Theta_1(t_i))-\wp_i(\Theta_2(t_i))\right\|+C_1\int_0^t\widehat{C}_2e^{-\gamma\hbar}\times$$

$$\left[\left\|\sigma_{\tilde{\Theta}_1}(d_1(\hbar))-\sigma_{\tilde{\Theta}_2}(d_1(\hbar))\right\|+\ldots+\left\|\sigma_{\tilde{\Theta}_1}(d_n(\hbar))-\sigma_{\tilde{\Theta}_2}(d_n(\hbar))\right\|\right.$$

$$\left.+\left\|\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}_1}(d_{n+1}(\Im))\right)d\Im-\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}_2}(d_{n+1}(\Im))\right)d\Im\right\|\right]d\hbar$$

$$+C_1\int_0^t\widehat{C}_3e^{-\gamma\hbar}\left[\left\|\sigma_{\tilde{\Theta}_1}(\varphi_1(\hbar))-\sigma_{\tilde{\Theta}_2}(\varphi_1(\hbar))\right\|+\ldots+\left\|\sigma_{\tilde{\Theta}_1}(\varphi_m(\hbar))-\sigma_{\tilde{\Theta}_2}(\varphi_m(\hbar))\right\|\right.$$

$$\left.+\left\|\int_0^{\hbar}J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}_1}(\varphi_{m+1}(\Im))\right)d\Im-J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}_2}(\varphi_{m+1}(\Im))\right)d\Im\right\|\right]d\hbar$$

$$\le C_1\left\|\varsigma(\tilde{\Theta}_1)-\varsigma(\tilde{\Theta}_2)\right\|+C_1\sum_{i=1}^{k}\left\|\wp_i(\Theta_1(t_i))-\wp_i(\Theta_2(t_i))\right\|+C_1\widehat{C}_2\int_0^t e^{-\gamma\hbar}\times$$

$$\left[\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|+\ldots+\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|+C_4\int_0^{\hbar}\left\|\sigma_{\tilde{\Theta}_1}(d_{n+1}(\Im))-\sigma_{\tilde{\Theta}_2}(d_{n+1}(\Im))\right\|d\Im\right]d\hbar$$

$$+C_1\widehat{C}_3\int_0^t e^{-\gamma\hbar}\left[\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|+\ldots+\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|\right.$$

$$\left.+C_5\int_0^{\hbar}\left\|\sigma_{\tilde{\Theta}_1}(\varphi_{m+1}(\Im))-\sigma_{\tilde{\Theta}_2}(\varphi_{m+1}(\Im))\right\|d\Im\right]d\hbar$$

$$\le C_1\left\|\varsigma(\tilde{\Theta}_1)-\varsigma(\tilde{\Theta}_2)\right\|+C_1\sum_{i=1}^{k}\left\|\wp_i(\Theta_1(t_i))-\wp_i(\Theta_2(t_i))\right\|+C_1\widehat{C}_2\int_0^t e^{-\gamma\hbar}\times$$

$$\left[n\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|+C_4c\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|\right]d\hbar+C_1\widehat{C}_3\int_0^t e^{-\gamma\hbar}\left[m\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|+C_5c\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|\right]d\hbar$$

$$\le C_1\left\|\varsigma(\tilde{\Theta}_1)-\varsigma(\tilde{\Theta}_2)\right\|+C_1\sum_{i=1}^{k}\left\|\wp_i(\Theta_1(t_i))-\wp_i(\Theta_2(t_i))\right\|$$

$$+C_1\left[\left(\widehat{C}_2n+\widehat{C}_3m\right)+c\left(\widehat{C}_2C_4+\widehat{C}_3C_5\right)\right]\int_0^t e^{-\gamma\hbar}\left\|\sigma_{\tilde{\Theta}_1}(\hbar)-\sigma_{\tilde{\Theta}_2}(\hbar)\right\|d\hbar\,.$$

By using the Gronwall's inequality once more, we have that, for t, Θ_1, Θ_2 as defined above

$$e^{-\gamma t}\left\|\sigma_{\tilde{\Theta}_1}(t)-\sigma_{\tilde{\Theta}_2}(t)\right\|\le C_1\left[\left\|\varsigma(\tilde{\Theta}_1)-\varsigma(\tilde{\Theta}_2)\right\|+\sum_{i=1}^{k}\left\|\wp_i(\Theta_1(t_i))-\wp_i(\Theta_2(t_i))\right\|\right]\times e^{C_1\left[\left(\widehat{C}_2n+\widehat{C}_3m\right)+c\left(\widehat{C}_2C_4+\widehat{C}_3C_5\right)\right]c},\ \forall t\in[0,c].$$

This again leads to

$$\left\|I\Theta_1 - I\Theta_2\right\|_{PC} \le C_1\left[\left\|\varsigma\left(\tilde{\Theta}_1\right)-\varsigma\left(\tilde{\Theta}_2\right)\right\| + \sum_{i=1}^{k}\left\|\wp_i\left(\Theta_1\left(t_i\right)\right)-\wp_i\left(\Theta_2\left(t_i\right)\right)\right\|\right]\times$$

$e^{\left\{C_1\left[\left(\hat{C}_2 n+\hat{C}_3 m\right)+c\left(\hat{C}_2 C_4+\hat{C}_3 C_5\right)\right]+\gamma\right\}c}$, $\forall \varepsilon \le t \le c, \Theta_1, \Theta_2 \in C_{\lambda}\left(\varepsilon\right)$. Thus, the operator I is continuous.

Step 3. I is compact operator.

To proceed it, we assume the decomposition $I = I_1 + I_2$, where I_1 and I_2 are the operators on $C_{\lambda}\left(\varepsilon\right)$ defined respectively by

$$\left(I_1\Theta\right)(t) = F(t,0)\left[\omega_0 - \varsigma\left(\tilde{\Theta}\right)\right] + \int_0^t F(t,\hbar)$$

$$\times\left[P_1\left(\hbar, \sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right), \int_0^{\hbar} J_1\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right.$$

$$\left.+P_2\left(\hbar, \sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right), \int_0^{\hbar} J_2\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right]d\hbar, \ \varepsilon \le t \le c.$$

$$\left(I_2\Theta\right)(t) = \sum_{0<t_i<t} F\left(t, t_i\right)\wp_i\left(\Theta\left(t_i\right)\right), \ \varepsilon \le t \le c.$$

First of all we demonstrate that I_1 is compact operator.

(a) $I_1\left(C_{\lambda}\left(\varepsilon\right)\right)$ is equicontinuous.

Suppose that $\varepsilon \le t_1 < t_2 \le c$, we obtain

Consider $\Theta \in C_{\lambda}\left(\varepsilon\right)$, we get

$$\begin{aligned}
\left\|I_1\Theta\left(t_2\right) - I_1\Theta\left(t_1\right)\right\| \le & \left\|\left[F\left(t_2,0\right)-F\left(t_1,0\right)\right]\left[\omega_0 - \varsigma\left(\tilde{\Theta}\right)\right]\right\| + \left\|\int_0^{t_2} F\left(t_2,\hbar\right)\right. \\
& \times P_1\left(\hbar, \sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right), \int_0^{\hbar} J_1\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right) \\
& \left. - \int_0^{t_1} F\left(\lambda_1,\hbar\right)P_1\left(\hbar, \sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right), \int_0^{\hbar} J_1\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right\|d\hbar \\
& + \left\|\int_0^{t_2} F\left(t_2,\hbar\right)P_2\left(\hbar, \sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right), \int_0^{\hbar} J_2\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right. \\
& \left. - \int_0^{t_1} F\left(t_1,\hbar\right)P_2\left(\hbar, \sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right), \int_0^{\hbar} J_2\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right\|d\hbar \\
\le & \left\|F\left(t_2,0\right)-F\left(t_1,0\right)\right\|\left[\left\|\omega_0\right\| + \Lambda(t)\right] + \int_0^{t_1}\left\|F\left(t_2,\hbar\right)-F\left(t_1,\hbar\right)\right\| \\
& \times\left\|P_1\left(\hbar, \sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right), \int_0^{\hbar} J_1\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right\|d\hbar \\
& + C_1 e^{\gamma t_2}\int_{t_1}^{t_2} e^{-\gamma\hbar}\left\|P_1\left(\hbar, \sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right), \int_0^{\hbar} J_1\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right\|d\hbar \\
& + \int_0^{t_1}\left\|F\left(t_2,\hbar\right)-F\left(t_1,\hbar\right)\right\|\left\|P_2\left(\hbar, \sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right), \int_0^{\hbar} J_2\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right\|d\hbar \\
& + C_1 e^{\gamma t_2}\int_{t_1}^{t_2} e^{-\gamma\hbar}\left\|P_2\left(\hbar, \sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right), \int_0^{\hbar} J_2\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right\|d\hbar.
\end{aligned} \tag{8}$$

Noting that

$$\left\|P_1\left(\hbar, \sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right), \dots, \sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right), \int_0^{\hbar} J_1\left(\hbar, \Im, \sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right\|$$

$$\leq \Big\| P_1\Big(\hbar, \sigma_{\tilde{\Theta}}(d_1(\hbar)), ..., \sigma_{\tilde{\Theta}}(d_n(\hbar)), \int_0^{\hbar} J_1(\hbar, \Im, \sigma_{\tilde{\Theta}}(d_{n+1}(\Im)))d\Im\Big)$$

$$-P_1(\hbar, 0, ..., 0)\Big\| + \|P_1(\hbar, 0, ..., 0)\|$$

$$\leq \widehat{C}_2\left[\|\sigma_{\tilde{\Theta}}(d_1(\hbar))\| + ... + \|\sigma_{\tilde{\Theta}}(d_n(\hbar))\| + \left\|\int_0^{\hbar} J_1(\hbar, \Im, \sigma_{\tilde{\Theta}}(d_{n+1}(\Im)))d\Im\right\|\right] + \breve{C}_2$$

$$\leq \widehat{C}_2\left[\|\sigma_{\tilde{\Theta}}(\hbar)\| + ... + \|\sigma_{\tilde{\Theta}}(\hbar)\| + \int_0^{\hbar}\Big\{\|J_1(\hbar, \Im, \sigma_{\tilde{\Theta}}(\Im)) - J_1(\hbar, 0, 0)\| + \|J_1(\hbar, 0, 0)\|\Big\}d\Im\right] + \breve{C}_2$$

$$\leq \widehat{C}_2\left[n\|\sigma_{\tilde{\Theta}}(\hbar)\| + c\left\{C_4 \sup_{\hbar \in [\varepsilon, c]}\|\sigma_{\tilde{\Theta}}(\hbar)\| + \breve{C}_4\right\}\right] + \breve{C}_2$$

$$\leq \widehat{C}_2\left[(n + cC_4)t + c\breve{C}_4\right] + \breve{C}_2 .$$

and

$$\Big\| P_2\Big(\hbar, \sigma_{\tilde{\Theta}}(\varphi_1(\hbar)), ..., \sigma_{\tilde{\Theta}}(\varphi_m(\hbar)), \int_0^{\hbar} J_2(\hbar, \Im, \sigma_{\tilde{\Theta}}(\varphi_{m+1}(\Im)))d\Im\Big)\Big\|$$

$$\leq \Big\| P_2\Big(\hbar, \sigma_{\tilde{\Theta}}(\varphi_1(\hbar)), ..., \sigma_{\tilde{\Theta}}(\varphi_m(\hbar)), \int_0^{\hbar} J_2(\hbar, \Im, \sigma_{\tilde{\Theta}}(\varphi_{m+1}(\Im)))d\Im\Big)$$

$$-P_2(\hbar, 0, ..., 0)\Big\| + \|P_2(\hbar, 0, ..., 0)\|$$

$$\leq \widehat{C}_3\left[\|\sigma_{\tilde{\Theta}}(\varphi_1(\hbar))\| + ... + \|\sigma_{\tilde{\Theta}}(\varphi_m(\hbar))\| + \left\|\int_0^{\hbar} J_2(\hbar, \Im, \sigma_{\tilde{\Theta}}(\varphi_{m+1}(\Im)))d\Im\right\|\right] + \breve{C}_3$$

$$\leq \widehat{C}_3\left[\|\sigma_{\tilde{\Theta}}(\hbar)\| + ... + \|\sigma_{\tilde{\Theta}}(\hbar)\| + \int_0^{\hbar}\Big\{\|J_2(\hbar, \Im, \sigma_{\tilde{\Theta}}(\Im)) - J_2(\hbar, 0, 0)\| + \|J_2(\hbar, 0, 0)\|\Big\}d\Im\right] + \breve{C}_3$$

$$\leq \widehat{C}_3\left[m\|\sigma_{\tilde{\Theta}}(\hbar)\| + c\left\{C_5 \sup_{\hbar \in [\varepsilon, c]}\|\sigma_{\tilde{\Theta}}(\hbar)\| + \breve{C}_5\right\}\right] + \breve{C}_3$$

$$\leq \widehat{C}_3\left[(m + cC_5)t + c\breve{C}_5\right] + \breve{C}_3 .$$

After putting the above values in equation (8), we notice that $\|I_1\Theta(t_2) - I_1\Theta(t_1)\| \to 0$ independently of $\Theta \in C_{\lambda}(\varepsilon)$ as $t_2 - t_1 \to 0$. This follows from the fact that the compactness of $F(t,\hbar)$ for $t,\hbar > 0$ implies continuity in the uniform operator topology. Therefore, the family $\{(I_1\Theta) : \Theta \in C_{\lambda}(\varepsilon)\}$ is equi -continuous on $[\varepsilon, c]$.

(b) The set $I_1(C_{\lambda}(\varepsilon))(t)$ is precompact in V .

Set $\varepsilon < t \leq \hbar \leq c$ and suppose a real number ξ fulfilling $0 < \xi < t$. For $\Theta \in C_{\lambda}(\varepsilon)$, we have

$$(I_{1,\xi}\Theta)(t) = F(t,0)\left[\omega_0 - \varsigma(\tilde{\Theta})\right]$$

$$+\int_0^{t-\xi} F(t,\hbar) P_1\Big(\hbar, \sigma_{\tilde{\Theta}}(d_1(\hbar)), ..., \sigma_{\tilde{\Theta}}(d_n(\hbar)), \int_0^{\hbar} J_1(\hbar, \Im, \sigma_{\tilde{\Theta}}(d_{n+1}(\Im)))d\Im\Big)d\hbar$$

$$+\int_0^{t-\xi} F(t,\hbar) P_2\Big(\hbar, \sigma_{\tilde{\Theta}}(\varphi_1(\hbar)), ..., \sigma_{\tilde{\Theta}}(\varphi_m(\hbar)), \int_0^{\hbar} J_2(\hbar, \Im, \sigma_{\tilde{\Theta}}(\varphi_{m+1}(\Im)))d\Im\Big)d\hbar .$$

By applying the compactness of $F(t,\hbar)$ for $t,\hbar > 0$, it follows the set $\{(I_{1,\xi}\Theta)(t) : \Theta \in C_{\lambda}(\varepsilon)\}$ is precompact $\Theta \in C_{\lambda}(\varepsilon)$ for ξ , $0 < \xi < t$. Also, $\forall \Theta \in C_{\lambda}(\varepsilon)$, we obtain

$$\|(I_1\Theta)(t) - (I_{1,\xi}\Theta)(t)\|$$

$$\leq \int_{t-\xi}^{t}\Big\| F(t,\hbar)\Big[P_1\Big(\hbar, \sigma_{\tilde{\Theta}}(d_1(\hbar)), ..., \sigma_{\tilde{\Theta}}(d_n(\hbar)), \int_0^{\hbar} J_1(\hbar, \Im, \sigma_{\tilde{\Theta}}(d_{n+1}(\Im)))d\Im\Big)$$

$$+P_2\Big(\hbar, \sigma_{\tilde{\Theta}}(\varphi_1(\hbar)), ..., \sigma_{\tilde{\Theta}}(\varphi_m(\hbar)), \int_0^{\hbar} J_2(\hbar, \Im, \sigma_{\tilde{\Theta}}(\varphi_{m+1}(\Im)))d\Im\Big)\Big]\Big\| d\hbar$$

$$\le C_1 e^{\gamma c}\int_{t-\xi}^{t} e^{-\gamma\hbar}\left\|\left[P_1\left(\hbar,\sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right),\int_0^{\hbar} J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right.\right.$$

$$\left.\left.+P_2\left(\hbar,\sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right),\int_0^{\hbar} J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right]\right\|d\hbar$$

$$\le C_1 e^{\gamma c}\int_{t-\xi}^{t} e^{-\gamma\hbar}\left[\hat{C}_2\left\{\left(n+cC_4\right)t+c\breve{C}_4\right\}+\breve{C}_2+\hat{C}_3\left\{\left(m+cC_5\right)t+c\breve{C}_5\right\}+\breve{C}_3\right]d\hbar .$$

So, there are precompact sets arbitrarily close to the set $\left\{\left(I_1\Theta\right):\Theta\in C_{\lambda}(\varepsilon)\right\}$. Consequently, the set $\left\{\left(I_1\Theta\right):\Theta\in C_{\lambda}(\varepsilon)\right\}$ is precompact in V.

Further, it remains to check that I_2 is also a compact operator. From [43], we see that I_2 is compact operator. So, we omitted here the proof of this part. Hence I is a compact operator.

Step 4. Now, our aim is to establish that $\exists$ an open set $S\subseteq PC_{\varepsilon}$ with $\Theta\notin\nu I\Theta$ for $0<\nu<1$ and $\Theta\in\partial S$. Consider $\nu\in(0,1)$ $\Theta\in PC_{\varepsilon}$ be a potential solution of $\Theta=\nu I\Theta$ for some $0<\nu<1$. Hence, $\forall 0<t\le c$,

$$\Theta(t)=\nu\sigma_{\tilde{\Theta}}(t)=\nu F(t,0)\left[\omega_0-\varsigma\left(\tilde{\Theta}\right)\right]+\nu\sum_{0<t_i<t}F\left(t,t_i\right)\wp_i\left(\Theta\left(t_i\right)\right)$$

$$+\nu\int_0^t F(t,\hbar)\left[P_1\left(\hbar,\sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right),\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right. \tag{9}$$

$$\left.+P_2\left(\hbar,\sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right),\int_0^{\hbar}J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right]d\hbar .$$

From assumptions $(A_1)-(A_5)$ and $\forall t\in\Xi$, we obtain $\left\|\Theta(t)\right\|\le\left\|\sigma_{\tilde{\Theta}}(t)\right\|$ and

$$e^{-\gamma t}\left\|\sigma_{\tilde{\Theta}}(t)\right\|\le C_1\left[\left\|\omega_0+\varsigma\left(\tilde{\Theta}\right)\right\|\right]+C_1\sum_{i=1}^{k}\rho_i\left\|\Theta\left(t_i\right)\right\|$$

$$+C_1\int_0^t e^{-\gamma\hbar}\left[P_1\left(\hbar,\sigma_{\tilde{\Theta}}\left(d_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(d_n(\hbar)\right),\int_0^{\hbar}J_1\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(d_{n+1}(\Im)\right)\right)d\Im\right)\right.$$

$$\left.+P_2\left(\hbar,\sigma_{\tilde{\Theta}}\left(\varphi_1(\hbar)\right),...,\sigma_{\tilde{\Theta}}\left(\varphi_m(\hbar)\right),\int_0^{\hbar}J_2\left(\hbar,\Im,\sigma_{\tilde{\Theta}}\left(\varphi_{m+1}(\Im)\right)\right)d\Im\right)\right]d\hbar$$

$$\le C_1\left[\left\|\omega_0\right\|+\Lambda\left\|\left(\tilde{\Theta}\right)\right\|+\sum_{i=1}^{k}\rho_i\left\|\Theta\right\|_{PC}\right]+\frac{C_1\left[c\left(\hat{C}_2\breve{C}_4+\hat{C}_3\breve{C}_5\right)+\left(\breve{C}_2+\breve{C}_3\right)\right]}{\gamma}$$

$$+C_1\left[\left(\hat{C}_2 n+\hat{C}_3 m\right)+c\left(\hat{C}_2 C_4+\hat{C}_3 C_5\right)\right]\int_0^t e^{-\gamma\hbar}\left\|\sigma_{\tilde{\Theta}}(\hbar)\right\|d\hbar .$$

With the assistance of Gronwall's inequality, the following outcome has obtained

$$e^{-\gamma t}\left\|\sigma_{\tilde{\Theta}}(t)\right\|\le C_1\left[\left\|\omega_0\right\|+\Lambda\left\|\left(\tilde{\Theta}\right)\right\|+\sum_{i=1}^{k}\rho_i\left\|\Theta\right\|_{PC}\right]+\frac{C_1\left[c\left(\hat{C}_2\breve{C}_4+\hat{C}_3\breve{C}_5\right)+\left(\breve{C}_2+\breve{C}_3\right)\right]}{\gamma}\times\ e^{C_1\left[\left(\hat{C}_2 n+\hat{C}_3 m\right)+c\left(\hat{C}_2 C_4+\hat{C}_3 C_5\right)\right]c} .$$

As a result,

$$\left\|\Theta\right\|_{PC}\le C_1\left[\left\|\omega_0\right\|+\Lambda\left\|\left(\tilde{\Theta}\right)\right\|+\sum_{i=1}^{k}\rho_i\left\|\Theta\right\|_{PC}\right]+\frac{C_1\left[c\left(\hat{C}_2\breve{C}_4+\hat{C}_3\breve{C}_5\right)+\left(\breve{C}_2+\breve{C}_3\right)\right]}{\gamma}\times\ e^{\left\{C_1\left[\left(\hat{C}_2 n+\hat{C}_3 m\right)+c\left(\hat{C}_2 C_4+\hat{C}_3 C_5\right)\right]+\gamma\right\}c},$$

and therefore

$$\frac{\left\|\Theta\right\|_{PC}}{C_1\left[\left\|\omega_0\right\|+\Lambda\left\|\left(\tilde{\Theta}\right)\right\|+\sum_{i=1}^{k}\rho_i\left\|\Theta\right\|_{PC}+\frac{\left\{c\left(\hat{C}_2\breve{C}_4+\hat{C}_3\breve{C}_5\right)+\left(\breve{C}_2+\breve{C}_3\right)\right\}}{\gamma}\right]D_0}\le 1 .$$

From assumption (A_6), $\exists$ a constant Q^* in such a way that $\left\|\Theta\right\|_{PC}\ne Q^*$. Set

$S=\left\{\Theta\in PC\left(\left[\varepsilon,c\right],V\right);\sup_{\varepsilon\le t\le c}\left\|\Theta\left(t\right)\right\|<Q^{*}\right\}.$

As a result of Steps 1-4 in Theorem 1, along with the use of Arzela-Ascoli theorem it is enough to demonstrate that $I:\overline{S}\to PC_{\varepsilon}$ is a compact map.

Since S is selected so that there is no $\omega\in\partial S$ in such a way that $\Theta\in\nu I\Theta$ for $\nu\in\left(0,1\right)$. So, in view of Lemma 3, we deduce that I has a fixed point $\tilde{\Theta}_{*}\in\overline{S}$. Consider $\omega=\sigma_{\tilde{\Theta}_{*}}$. Then, we get

$$\begin{aligned}\omega\left(t\right)&=F\left(t,0\right)\left[\omega_{0}-\varsigma\left(\tilde{\Theta}_{*}\right)\right]+\sum_{0<t_{i}<t}F\left(t,t_{i}\right)\wp_{i}\left(\Theta\left(t_{i}\right)\right)\\&+\int_{0}^{t}F\left(t,\hbar\right)\left[P_{1}\left(\hbar,\omega\left(d_{1}\left(\hbar\right)\right),...,\omega\left(d_{n}\left(\hbar\right)\right),\int_{0}^{\hbar}J_{1}\left(\hbar,\Im,\omega\left(d_{n+1}\left(\Im\right)\right)\right)d\Im\right)\right.\\&\left.+P_{2}\left(\hbar,\omega\left(\varphi_{1}\left(\hbar\right)\right),...,\omega\left(\varphi_{m}\left(\hbar\right)\right),\int_{0}^{\hbar}J_{2}\left(\hbar,\Im,\omega\left(\varphi_{m+1}\left(\Im\right)\right)\right)d\Im\right)\right]d\hbar.\end{aligned}\tag{10}$$

Noting that $\omega=\sigma_{\tilde{\Theta}_{*}}=\left(I\tilde{\Theta}_{*}\right)\left(t\right)=\tilde{\Theta}_{*},\varepsilon\le t\le c$. By $\left(A_{5}\right)$, we get

$\varsigma\left(\omega\right)=\varsigma\left(\tilde{\Theta}_{*}\right).$

In the insight of above discussion, it follows that the operator ω is W has a fixed point in the set $\overline{S}\subset PC\left(\Xi,V\right)$. Consequently, the problem (1)-(3) has a mild solution. Thus it establish the validity of Theorem 1.

4. Example

To explicitly reveal the analytical outcomes got earlier, we now state two illustrate example.

Example 1. Consider the following partial integrodifferential equation incorporating time varying delay with impulsive and nonlocal condition

$$\begin{aligned}\frac{\partial z\left(t,\omega\right)}{\partial t}&=\frac{\partial^{2}}{\partial\omega^{2}}\left[b_{0}\left(t\right)z\left(t,\omega\right)+\int_{0}^{t}T\left(t,\alpha\right)z\left(\alpha,\omega\right)\right]\\&+b_{1}\left(t\right)z\left(\sin t,\omega\right)+\sin z\left(t,\omega\right)+\frac{1}{1+t^{2}}\int_{0}^{t}b_{2}\left(\alpha\right)z\left(\sin\alpha,\omega\right)d\alpha\\&+\tilde{b}_{1}\left(t\right)z\left(\sin t,\omega\right)+\sin z\left(t,\omega\right)+\frac{1}{1+t^{2}}\int_{0}^{t}\tilde{b}_{2}\left(\alpha\right)z\left(\sin\alpha,\omega\right)d\alpha,\end{aligned}\tag{11}$$

$$\Delta z\left(t_{i},\omega\right)=\int_{0}^{t_{i}}e_{i}\left(t_{i}-\alpha\right)z\left(\alpha,\omega\right)d\omega,i=1,2,...,k,\tag{12}$$

$$z\left(t,0\right)=z\left(t,\pi\right)=0,\tag{13}$$

$$z\left(0,\omega\right)+\int_{\varepsilon}^{1}\left[z\left(\alpha,\omega\right)+\log\left(1+\left|z\left(\alpha,\omega\right)\right|\right)\right]d\alpha=z_{0}\left(\omega\right),t\in\left[0,1\right],\omega\in\left[0,\pi\right],\tag{14}$$

where $\varepsilon>0$, $z_{0}\left(\omega\right)\in V=L^{2}\left(\left[0,\pi\right]\right)$ and $z_{0}\left(0\right)=z_{0}\left(\pi\right)=0$. Here, the coefficients $b_{0}\left(t\right)$ is continuous on $0\le t\le c$ and the kernel $T\left(t,\alpha\right)$ is continuous for $0\le\alpha\le t\le c$.

Consider $V=L^{2}\left(\left[0,\pi\right]\right)$. We define the operator $A\left(t\right)$ by

$A\left(t\right)u=b_{0}\left(t\right)u''$

with domain $D\left(A\right)=\left\{u\in V:u,u''\text{ are absolutely continuous, }u''\in V,u\left(0\right)=u\left(1\right)=0\right\}$.

With these choices, $A\left(t\right)$ generates an evolution system $F\left(t,\alpha\right)$. This operator $F\left(t,\alpha\right)$ is constructed from the evolution system framework [35,44] and possess the property of compactness, fulfilling $\left\|F\left(t,\alpha\right)\right\|\le C_{1}e^{\gamma\left(t-\alpha\right)}$ where C_{1} and γ are some constants.

Moreover, we impose the assumptions that are stated below:

(a) The functions $b_i(.)$ and $\tilde{b}_i, i=1,2$ are continuous on the interval $[0,1]$. Also, the supremum norms of these remains strictly below unity, that is, $l_i = \sup_{0\le\alpha\le1}|b_i(\alpha)|<1, i=1,2$ and $\tilde{l}_i = \sup_{0\le\alpha\le1}|\tilde{b}_i(\alpha)|<1, i=1,2$

(b) $\forall i=1,2,...,k$, the functions $e_i : R\to R, i=1,2,...,k$, are continuous as well as bounded. Further, they are square integrable over the related domain, namely

$\tau_i = \left(\int_0^\pi (e_i(\alpha))^2 \, d\alpha\right)^{1/2} < \infty, \forall i=1,2,...,k$.

Define the function $P_1:[0,1]\times V\times V\to V, P_2:[0,1]\times V\times V\to V$, $J_1, J_2:[0,1]\times[0,1]\times V\to V$, $\wp_i : V\to V$ and $t: PC([0,1],V)\to V$ by

$$P_1\left(t, z(d(t)), \int_0^t J_1(t,\alpha, z(d(\alpha)))d\alpha\right)(\omega)$$

$$= b_1(t) z(\sin t,\omega) + \sin z(t,\omega) + \frac{1}{1+t^2}\int_0^t b_2(\alpha) z(\sin\alpha,\omega)d\alpha,$$

$$P_2\left(t, z(\beta(t)), \int_0^t J_2(t,\alpha, z(\beta(\alpha)))d\alpha\right)(\omega)$$

$$= \tilde{b}_1(t) z(\sin t,\omega) + \sin z(t,\omega) + \frac{1}{1+t^2}\int_0^t \tilde{b}_2(\alpha) z(\sin\alpha,\omega)d\alpha,$$

$$\int_0^t J_1(t,\alpha, z(d(\alpha)))(\omega)d\alpha = \frac{1}{1+t^2}\int_0^t b_2(\alpha) z(\sin\alpha,\omega)d\alpha,$$

$$\int_0^t J_2(t,\alpha, z(\beta(\alpha)))(\omega)d\alpha = \frac{1}{1+t^2}\int_0^t \tilde{b}_2(\alpha) z(\sin\alpha,\omega)d\alpha,$$

and

$$t(z)(\omega) = \int_\varepsilon^1 \left[z(\alpha,\omega) + \log(1+|z(\alpha,\omega)|)\right]d\alpha, z\in PC([0,1],V).$$

Then, equation (11)-(14) reduces to the abstract form (1)-(3). Since all the conditions of Theorem 1 are fulfilled, it follows that the problem (11)-(14) has a mild solution on $[0,1]$.

Example 2. Diffusive Population Model with Memory, Nonlocal Birth, Delay and Impulse.

Consider a population/Chemical model with diffusion as follows

$$\frac{\partial}{\partial t} z(\omega,t) = T^*\Delta\left(z(.,t) + \int_0^t k(t-\hbar) z(.,\hbar)d\hbar\right) + \left(a(t,\omega) - b(t,\omega) z(\omega, t-\tau(t))\right)$$
$$+\int_\Omega m(\omega,y) z(y, t-\sigma(t))dy, \omega\in\Omega=(0,\pi), t\in(0,c], t\ne t_i, \tag{15}$$

$$\Delta z(., t_i) = \wp_i\left(z(., t_i^-)\right), i=1,2,...,k, \tag{16}$$

$$z(.,0) = \phi(.) + \int_0^c \gamma(\hbar) z(.,\hbar)d\hbar, \tag{17}$$

together homogenous Dirichlet boundary conditions $z(0,t) = z(\pi,t) = 0$, where

$T^*>0$ is the diffusion constant and Δ is the Laplacian in ω

Memory kernel $k\in C([0,c])$

Delay functions $\tau,\sigma\in C([0,c])$ with $0\le\tau(t),\sigma(t)\le\tau_{\max}<c$.

$a,b\in C([0,c]\times\overline{\Omega})$ with a bounded and b nonnegative and bounded.

Spatial birth kernel $m\in L^\infty(\Omega\times\Omega)$. The spatial nonlocal birth is

$$\int_{\Omega} m(\omega, y) z(y, t-\sigma(t)) dy .$$

Impulse operator $\wp_i \in L(V)$

Nonlocal initial operator $\gamma \in C([0,c])$ and $\phi \in V$. Let $V = L^2(\Omega)$ and set $\omega(t) = z(.,t) \in V$.

By taking the appropriate choices, we can find the abstract form for (15)-(17) with homogenous Dirichlet boundary conditions and this abstract form is like (1)-(3) if P_2 and J_2 are zero. Under the given Assumptions which is enlisted in Section 2, the NIFIETVD admits at least one mild solution on $[0,c]$.

5. Conclusion

In the presented paper, we have addressed a new class of NIFIETVD. We have derived a criteria for existence of mild solution of NIFIETVD with impulsive effects and nonlocal conditions in Banach spaces by imposing the framework of resolvent operator in combination with fixed point theory and Leray-Schauder nonlinear alternative. Our analysis relax the usual compactness. The Lipschitz assumptions applied on the nonlocal term. Due to it the range of applicability of theory is extended. The outcome that is derived in this research work contribute to the growing theory of nonlocal IE with impulse effects and generate a basis of further research work. The results which are derived may be extended and generalized with stochastic effects and also to establish the stability solution for the same class. Further, in future, the results which are established in (1)–(3) and (11)–(14) can be used to find the numerical solution of such type of equation.

Conflict of Interest

The authors declare they have no conflicts of interest.

Generative AI Statement

The authors declare that no Gen AI was used in the creation of this manuscript.

References

[1] Hale JK. Theory of functional differential equations. Applied Mathematical Sciences, 1977. DOI: 10.1007/978-1-4612-9892-2

[2] Kumar K, Kumar R. A discussion on controllability of semilinear impulsive functional differential equations of second order. TWMS Journal of Applied and Engineering Mathematics, 2025, 15(3), 590-601.

[3] Kumar K, Kumar R. Existence of mild solutions for semilinear impulsive functional mixed integro-differential equations with nonlocal conditions. Applications and Applied Mathematics: An International Journal (AAM), 2014, 9(2), 659-671.

[4] Qumami NHM, Jain RS, Reddy BS. Mild solution for nonlocal impulsive functional fuzzy nonlinear integro-differential equations. Fixed Point Theory and Algorithms for Sciences and Engineering, 2025, 2025(1), 12. DOI: 10.1186/s13663-024-00779-w

[5] Bohner M, Tunç O, Tunç C. Existence of solutions for nonlinear impulsive multiple retarded differential and impulsive integro-differential equations of second order. Journal of Nonlinear and Convex Analysis, 2024, 25(9), 2337-2360.

[6] Samoilenko AM, Perestyuk NA. Impulsive differential equations. World Scientific, Singapore, 1995. DOI: 10.1142/2892

[7] Agarwal RP, Leiva H, Riera L, Lalvay S. Existence of solutions for impulsive neutral semilinear evolution equations with nonlocal conditions. Discontinuity, Nonlinearity, and Complexity, 2024, 13(2), 333-349. DOI: 10.5890/DNC.2024.06.011

[8] Karthikeyan K, Nieto JJ. Impulsive functional-abstract neutral functional differential nonlocal Cauchylinebreak problem with state-dependent delay, Discontinuity, Nonlinearity, and Complexity, 2022, 11(3), 387-393. DOI: 10.5890/DNC.2022.09.002

[9] Paul T, Anguraj A. Existence and uniqueness of nonlinear impulsive integro-differential equations. Discrete and Continuous Dynamical Systems-B, 2006, 6(5), 1191-1198. DOI: 10.3934/dcdsb.2006.6.1191

[10] Arora S, Mohan MT, Dabas J. Existence and approximate controllability of non-autonomous functional impulsive evolution inclusions in Banach spaces. Journal of Differential Equations, 2022, 307, 83-113. DOI: 10.1016/j.jde.2021.10.049

[11] Fall M, Didiya MD, Gnonlonfoun AW, Diop MA. On impulsive integrodifferential equations with state-dependent delay. Journal of Numerical Mathematics and Stochastics, 2022, 13(1), 29-54.

[12] Lakshmikantham V. Theory of integro-differential equations. CRC press, 1995.

[13] Wilson HR, Cowan JD. Excitatory and inhibitory interactions in localized populations of model neurons. Biophysical Journal, 1972, 12(1), 1-24. DOI: 10.1016/S0006-3495(72)86068-5

[14] Martin R, Chappell D, Chuzhanova N, Crofts JJ. A numerical simulation of neural felds on curved geometries. Journal of Computational Neuroscience, 2018, 45(2), 133-145. DOI: 10.1007/s10827-018-0697-5

[15] Medlock J, Kot M. Spreading disease: integrodifferential equations old and new. Mathematical Biosciences, 2003, 184(2), 201-222. DOI: 10.1016/S0025-5564(03)00041-5

[16] Balachandran K, Annapoorani N. Existence results for impulsive neutral evolution integrodifferential equations with infinite delay. Nonlinear Analysis: Hybrid Systems, 2009, 3(4), 674-684. DOI: 10.1016/j.nahs.2009.06.004

[17] Ouahab A. Local and global existence and uniqueness results for impulsive functional differential equations with multiple delay. Journal of Mathematical Analysis and Applications, 2006, 323(1), 456-472. DOI: 10.1016/j.jmaa.2005.10.015

[18] Wang J, Wei W. A Class of nonlocal impulsive problems for integrodifferential equations in Banach Spaces. Results in Mathematics/Resultate der Mathematik, 2010, 58, 379-397. DOI: 10.1007/s00025-010-0057-x
[19] Karthikeyan K, Anguraj A, Malar K, Trujillo JJ. Existence of mild and classical solutions for nonlocal impulsive integrodifferential equations in banach spaces with measure of noncompactness. International Journal of Differential Equations, 2014, 2014(1), 319250. DOI: 10.1155/2014/319250
[20] Ahmad B, Malar K, Karthikeyan K. A study of nonlocal problems of impulsive integrodifferential equations with measure of noncompactness. Advances in Difference equations, 2013, 2013(1),205. DOI: 10.1186/1687-1847-2013-205
[21] Byszewski L, Lakshmikantham V. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. Applicable analysis, 1991, 40(1), 11-19. DOI: 10.1080/00036819008839989
[22] Byszewski, L. Strong maximum principles for parabolic nonlinear problems with non-local inequalities together with arbitrary functions. Journal of mathematical analysis and applications, 1991, 156(2), 457-470. DOI: 10.1016/0022-247X(91)90409-S
[23] Lin Y, Liu JH. Semilinear integrodifferential equations with nonlocal Cauchy problem. Nonlinear Analysis: Theory, Methods & Applications, 1996, 26(5), 1023-1033. DOI: 10.1016/0362-546X(94)00141-0
[24] Ezzinbi K, Ghnimi S. Existence and regularity of solutions for some partial integrodifferential equations involving the nonlocal conditions. Numerical Functional Analysis and Optimization, 2019, 40(13), 1532-1549. DOI: 10.1080/01630563.2019.1604547
[25] Ezzinbi K, Ghnimi S, Taoudi, MA. Existence results for some partial integrodifferential equations with nonlocal conditions, Glasnik Matematicki, 2016, 51(2), 413-430. DOI: 10.3336/gm.51.2.09
[26] Tunç C, Tunç O. On the qualitative behaviors of Volterra-Fredholm integro differential equations with multiple time-varying delays. Arab Journal of Basic and Applied Sciences, 2024, 31(1), 440-453. DOI: 10.1080/25765299.2024.2386737
[27] Rezoug N, Salim A, Benchohra M. Nonlocal second order delay semilinear integro-differential time varying evolution equation with convex-power condensing operators. Filomat, 2024, 38(11), 3861-3880. DOI: 10.2298/FIL2411861R
[28] Akiladevi KS, Balachandran K, Kim D. On fractional time-varying delay integrodifferential equations with multi-point multi-term nonlocal boundary conditions. Nonlinear Functional Analysis and Applications, 2024, 29(3), 803-823. DOI: /10.22771/nfaa.2024.29.03.10
[29] Diallo MA, Ezzinbi K, Séne A. Impulsive integro-differential equations with nonlocal conditions in Banach spaces. Transactions of A. Razmadze Mathematical Institute, 2017, 171(3), 304-315. DOI: 10.1016/j.trmi.2016.12.002
[30] Hu B, Qiu Y, Zhou W, Zhu L. Existence of solution for an impulsive differential system with improved boundary value conditions. AIMS Mathematics, 2023, 8(7), 17197-17207. DOI: 10.3934/math.2023878
[31] Kumar K, Karnatak M, Kumar R. Nonlinear integrodifferential equations with time varying delay. Advances in the Theory of Nonlinear Analysis and its Applications, 2021, 5(3), 433-444. DOI: 10.31197/atnaa.814109
[32] Chadha A, Pandey DN. Existence of a mild solution for an impulsive nonlocal non-autonomous neutral functional differential equation. Annali Dell'Universita' di Ferrara, 2016, 62, 1-21. DOI: 10.1007/s11565-016-0242-2
[33] Balachandran K, Kumar RR. Existence of solutions of integrodifferential evolution equations with time varying delays. Applied Mathematics E-Notes, 2007, 7, 1-8.
[34] Grimmer R, Liu JH. Integrated semigroups and integrodifferential equations. Semigroup Forum, 1994, 48, 79-95. DOI: 10.1007/BF02573656
[35] Pruss J. On resolvent operators for linear integrodifferential equations of Volterra type. The Journal of Integral Equations, 1983, 5, 211-236.
[36] Oka H. Integrated resolvent operator. The Journal of Integral Equations and Applications, 1995, 7(2), 193-232. DOI: 10.1216/jiea/1181075869
[37] Freidman A. Partial differential equations. Holt, Rinehat and Winston, New York, 1969.
[38] Fitzgibbon WE. Semilinear functional equations in Banach spaces. Journal of Differential Equations, 1978, 29(1), 1-14. DOI: 10.1016/0022-0396(78)90037-2
[39] Dugundji J, Granas A. Fixed point theory. Monografie Matematyczne, 1982, 61.
[40] Pazy A. Semigroup of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 1983, 44.
[41] Zhu J. Existence and uniqueness of global solutions for non-autonomous evolution equations with state-dependent nonlocal conditions. Filomat, 2023, 37(3), 745-759. DOI: 10.2298/FIL2303745Z
[42] Elghandouri M, Ezzinbi K. Approximation of mild solutions of delay integro-differential equations on Banach spaces. Evolution Equations and Control Theory, 2023, 12(6), 1629-1657.
[43] Yan Z. Existence for a nonlinear impulsive functional integrodifferential equation with nonlocal conditions in Banach Spaces. Journal of Applied Mathematics & Informatics, 2011, 29 (3), 681-696.
[44] Khalil H, Zada A, Rhaima M, Popa IL. Existence and stability of neutral stochastic impulsive and delayed integro-differential system via resolvent operator. Fractal and Fractional, 2024, 8(11), 659. DOI: 10.3390/fractalfract8110659