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Generalized Reduced Gröbner Basis and Initial Ideal of Binomial Edge Ideal of Different Classes of Graphs

Ali Raza Shafiq[1], Sidra Bashir[2], Ilyas Khan[3,*]

[1]Department of Mathematics, University of Management and Technology, Lahore, Pakistan

[2]Department of Mathematics, Riphah International University, Islamabad, Pakistan

[3]Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan

[*]Corresponding author: Ilyas Khan, ilyas.khan_22@sms.edu.pk

Abstract

In this paper, we study the binomial edge ideals associated with three specific classes of graphs: the comb graph, the cross-ladder graph, and the n-sunlet graph. These graph structures offer a rich interplay between combinatorics and algebra, particularly in the context of Gröbner basis theory. For each graph, we explicitly compute the reduced Gröbner basis of the corresponding binomial edge ideal with respect to a lexicographic monomial order. Our computations involve a detailed analysis of admissible paths in the graphs, which play a central role in characterizing the generators of the Gröbner basis. Furthermore, we determine the initial ideals associated with each class and describe the families of monomials that arise in their minimal generating sets. The construction of these Gröbner bases not only offers insight into the structural properties of the respective graphs but also enables potential applications in algebraic statistics, computational algebra, and ideal theory. By classifying the admissible paths and systematically generating the Gröbner basis elements, our work provides a constructive and combinatorially motivated framework for understanding binomial edge ideals. These results contribute to the growing body of literature exploring the connections between graph-theoretic structures and algebraic invariants, and they open avenues for further investigations in more generalized or complex graph families.

Keywords

Monomial, Gröbner basis, Binomial edge ideal

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1. Introduction

Algebraic geometry is one of the important branches of modern mathematics that deals with the zeros of multivariable polynomials. Commutative algebra is the main tool used to study algebraic geometry. The homological features of current commutative algebra have become a recent and significant field of research, initiated by Melvin Hochster [1]. In 1975, Stanley used the concept of Cohen–Macaulay rings to prove the upper bound conjecture related to spheres. This generated further new developments in commutative algebra, leading to the field known as combinatorial commutative algebra, in which basic methods of commutative algebra are applied to study combinatorial objects such as simplicial complexes and convex polytopes. Stanley was the first to systematically apply these ideas and procedures from commutative algebra to discuss simplicial complexes through Stanley–Reisner rings, whose defining ideals consist of square-free monomials. Since then, the study of square-free monomial ideals, from both combinatorial and algebraic perspectives, has become an active area of research in commutative algebra [2]. In 1990, Villarreal associated monomial ideals with graphs, introducing what is now known as the monomial edge ideal [3]. In 2010, a new development in this subject appeared with the introduction of binomial edge ideals, independently by Herzog et al. [4] and Ohtani [5].

In computational algebra, a reduced Gröbner basis is a special kind of generating set of an ideal in a polynomial ring, and solving a system of polynomials using the reduced Gröbner basis is one of its remarkable applications. The Gröbner bases of binomial edge ideals were studied in [5] and [4]. In particular, those graphs were characterized for which the generators of an ideal form a Gröbner basis with respect to the lexicographic order. These graphs are called closed graphs. Crupi et al. [6] characterized all binomial edge ideals with quadratic Gröbner bases with respect to the lexicographic order. Badaini et al. [7] studied the universal Gröbner basis of binomial edge ideals. Zafar et al. [8] studied the initial ideals in degree 2 of binomial edge ideals for different classes of graphs. For more details on binomial edge ideals, we refer to [9-18]. Recent developments have significantly advanced the study of binomial edge ideals through various algebraic and combinatorial approaches. In particular, the regularity and depth of generalized binomial edge ideals have been explored in [19,20], while regularity bounds and their sharpness across different graph classes have been addressed in [21]. Structural properties such as the sequential Cohen–Macaulayness of binomial edge ideals in cycles and wheels are examined in [22]. Moreover, the binomial edge ideals associated with Levi graphs from curve arrangements are studied in [23], offering new perspectives on their algebraic behavior. These contributions complement the foundational results and provide a broader context for analyzing Gröbner bases and initial ideals of binomial edge ideals in various graph structures.

Over the past few years, there has been growing interest in the algebraic and homological properties of binomial edge ideals, particularly in relation to their Gröbner bases and initial ideals. Jayanthan et al. [24] investigated partial Betti splittings, providing a structured approach to analyzing minimal resolutions of binomial edge ideals. The regularity behavior of powers of such ideals, especially in multipartite settings, has been studied in detail by Wang and Tang [25]. On a broader spectrum, Stelzer [26] examined reduced Gröbner bases arising from matrix Schubert varieties, revealing combinatorial aspects relevant to binomial ideals. Bhardwaj and Saha [27] studied the regularity of Cohen–Macaulay binomial edge ideals, identifying new classes with desirable homological properties. Further contributions by Lax et al. [22] focused on the sequential Cohen–Macaulay property for specific graph families. Depth-related characteristics were analyzed by Anuvinda et al. [20], who addressed the structural depth of generalized binomial edge ideals. Complementing these studies, Kumar [28] investigated the Rees algebras and special fiber rings associated with binomial edge ideals, offering insights into their blow-up algebras.

Motivated by these developments, the present work aims to construct generalized reduced Gröbner bases and determine initial ideals for various classes of graphs, thereby extending the current theoretical framework. Many graph classes remain unexplored with respect to their Gröbner bases and initial ideals. Most of the existing literature focuses on simple graph families such as paths, cycles, and trees. However, graph structures with more complex connectivity, such as comb graphs, cross-ladder graphs, and sunlet graphs, pose unique combinatorial challenges and offer potential insights into the structural behavior of binomial edge ideals. Computing Gröbner bases and initial ideals for such graphs not only deepens our understanding of their algebraic invariants but also helps classify their homological properties, such as regularity, projective dimension, and Cohen–Macaulayness.

2. Preliminaries

The notation used in the article will be introduced in this section. We will discuss some important definitions of commutative algebra like monomial ideal, reduced Gröbner basis and binomial edge ideal.

Definition 1 ([29]). A ring R is a non-empty set equipped with two binary operations + and × such that: (R,+) is an abelian group, (R,×) is a semigroup, multiplication is distributive over addition from both sides:

$a\times(b+c)=a\times b+a\times c,\ (a+b)\times c=a\times c+b\times c,\ \forall a,b,c\in R.$

Example 1. Examples of rings include:

The set of integers Z, the set of rational numbers Q, the set of real numbers R, the set of all 2 × 2 matrices with real entries.

Definition 2 ([29]). A ring of the form: $R[x]=\{\sum_{k=0}^{n} a_k x^k \mid a_k \in R \wedge n \geq 0\}$, is called a polynomial ring. If $S=K[x_1, x_2, \ldots, x_n]$, then the ring in n-variables will be a polynomial ring.

Definition 3 ([29]). A surbing with the condition that $rs \in I$ for every $r \in R$ and for all $s \in I$ is called a ideal $I \subseteq R$.

Definition 4 ([29]). Consider the $x_1,\ldots,x_n$ be the indeterminates of the ring $S=K[x_1, \ldots, x_n]$. Then any product of the form: $x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}$, is called a monomial, where the exponents always occur as positive integers. The set of all monomials of S is denoted by mon (S).

Example 2 ([30]). The expression $x_1^2 x_2^3 x_4^5$ is an example of a monomial in the ring $S=K[x_1, x_2, x_3, x_4]$.

Definition 5 ([30]). Let $u=x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n} \in mon(S)$ then degree of u is denoted by Deg(u) and it is equal to sum of all exponents, i.e. $Deg(u)=c_1+\ldots+c_n$.

Definition 6 ([31]). A monomial ideal is an ideal I created by monomials such that $I \subset S$.

Example 3. The ideal $I=\langle x_1^2, x_3x_4, x_4^3 \rangle$ is an example of a monomial ideal in the ring $K[x_1, x_2, x_3, x_4]$.

2.1 Monomial Order

Definition 7 ([30]). A monomial ordering on $K[x_1, \ldots, x_n]$ is total order "<'' on mon(S) such that:

For all mon(S), $1<u$.

$uw<vw$ for all $w \in mon(S)$ if $u, v \in mon(S)$ such that $u<v$.

Definition 8 ([30]). (Pure lexicographic order):

Consider $u=x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $v=x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \in mon(S)$

then we say, $u>_{plex}v$

if in vector difference $(a_1, a_2, \ldots, a_n)-(b_1, b_2, \ldots, b_n) \in Z^n \geq 0$ the left most non-zero entry is positive, where $Z^n=\{(a_1, a_2, \ldots, a_n) | a_i \in Z\}$.

Definition 9 ([30]). (Lexicographic order) Let $u=x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $v=x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ be monomials in $S=K[x_1, x_2, \ldots, x_n]$. We say that

$u>_{lex}v$,

if one of the following holds:

$deg(u)>deg(v)$,

where $deg(u)=a_1+a_2+\ldots+a_n$,

or $deg(u)=deg(v)$ and $u>_{plex}v$, where $>_{plex}$ denotes the pure lexicographic order (compare the exponents from left to right and take the first index where they differ).

This ordering is known as the degree lexicographic order.

Definition 10 ([31]). (Reverse lexicographic order) Consider $u=x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $v=x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \in mon(S)$ then we say $u>_{rlex}v$ if in vector difference $(a_1, a_2, \ldots, a_n)-(b_1, b_2, \ldots, b_n) \in Z^n \geq 0$ the right most non-zero entry is positive.

2.2 Gröbner Basis

In this section we will discuss some those definitions of commutative algebra which are related to main result of our theorems.

Definition 11 ([30]). Let $0 \neq g \in S$ then the set of all monomial in g is called support of g denoted by supp(g). We can say that $supp(g) \subseteq mon(S)$.

Definition 12 ([30]). To fix a monomial "<"on S, let $S=K[x_1,x_2,\ldots,x_n]$. The initial monomial of g, represented by $in_<(g)$, is the greatest monomial that appears in supp(g) with respect to "<", if $g \neq 0$. The leading term of g with respect to "<" is the product $c.in_<(g)$, and the leading coefficient of g with respect to "<" is the coefficient c of $in_<(g)$ in the expression of g, shown by lc(g).

Definition 13 ([30]). Let I be an ideal of S and "<" be monomial order on S. The initial ideal of S with regard to "<" is a monomial ideal generated by $\{in_<(g): g \in I-\{0\}\}$, denoted by $in_<(I)$. If $I=0$, then $in_<(I)=<0>$.

Definition 14 ([30]). Let $S=K[x_1, x_2, \ldots, x_n]$ be a polynomial ring over a field K, and let < be a monomial order on S. Let $I \subset S$ be a non-zero ideal. A finite set of non-zero polynomials $\{f_1, f_2, \ldots, f_s\} \subset I$ is called a Gröbner basis of I with respect to the monomial order < if

$in_<(I)=\langle in_<(f_1), in_<(f_2), \ldots, in_<(f_s)\rangle$,

where $in_<(f_i)$ denotes the leading term of f_i with respect to <, and $in_<(I)$ denotes the initial ideal of I with respect to"<".

The set of all Gröbner basis of I with respect to "<" is denoted by G(I).

Definition 15 ([30]). Let I is ideal of S and $G(I)=\{f_1, \ldots, f_s\}$ be a Gröbner basis with respect to monomial order "<" on S. Consequently, each Gröbner basis for I is a generator system for I. i.e. $I=<f_1, \ldots, f_s>$.

Definition 16 ([30]). If "<" is monomial ordering on S and I is ideal of S and $G(I)=\{f_1, \ldots, f_r\}$ be Gröbner basis of I. Next, under the "<", G(I) is decreased Gröbner basis of I if

(1) $lc(f)=1 \ \forall \ f \in G(I)$.

(2) For all $s \neq \theta$ no monomial in $supp(g_\theta)$ is divided by $in_<(g_s)$.

3. Binomial Edge Ideal

The significant findings on the binomial edge ideal of graphs are covered in this section. Let G be a simple graph with the vertex set $[n]=\{1, 2, \ldots, n\}$ in all definitions.

Definition 17 ([2]). The monomial edge ideal is an ideal denoted by I_G such that $I_G \subseteq S$. Such that

$$I_G = <x_s y_\theta : \{s, \theta\} \in E(G)> \quad (1)$$

where $s<\theta$.

Definition 18 ([4]). Let $G=(V(G), E(G))$ be a finite simple graph with vertex set $V(G)=\{1, 2, \ldots, n\}$ and edge set E(G). Let $S=K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring over a field K.

The binomial edge ideal of G, denoted by J_G, is the ideal in S defined as:

$$J_G = \langle f_{ij}=x_i y_j - x_j y_i \mid \{i, j\} \in E(G), i<j \rangle \quad (2)$$

Definition 19 ([4]). (Admissible Path) Let $G=(V(G), E(G))$ be a simple graph and let $i, j \in V(G)$ with $i<j$ under a fixed monomial order.

A path $P: i=i_0 \rightarrow i_1 \rightarrow \ldots \rightarrow i_r=j$ in G is called an admissible path from i to j if the following conditions are satisfied:

(1) The vertices $i_0, i_1, \ldots, i_r$ are all distinct, i.e., $i_k \neq i_l$ for all $k \neq l$.

(2) For each intermediate vertex i_k with $1 \leq k \leq r-1$, one has either $i_k<i$ or $i_k>j$ (with respect to the given monomial order).

(3) For any subset $\{j_1, \ldots, j_s\} \subset \{i_1, \ldots, i_{r-1}\}$, the induced subgraph on the vertex set $\{i, j_1, \ldots, j_s, j\}$ does not contain a path fromi to j other than P itself.

Definition 20 ([4]). (Associated Monomial) Let G be a simple graph with vertex set $\{1, 2, \ldots, n\}$, and let J_G be the binomial edge ideal in the polynomial ring $S=K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ over a field K.

Let $P: i=i_0 \rightarrow i_1 \rightarrow \ldots \rightarrow i_r=j$ be an admissible path in G with $i<j$. Then the associated monomial corresponding to the path P is denoted by α_P and is defined as:

$$\alpha_P = \prod_{k=1}^{r-1} x_{i_k} \quad (3)$$

If $f_{ij}=x_i y_j - x_j y_i \in J_G$, then the product:

$\alpha_P \times f_{ij} = \alpha_P (x_i y_j - x_j y_i)$, is called the binomial associated with the path P.

Definition 21 ([4]). Think about an admissible route. For $i<j$, $P: i_0 \rightarrow i_1 \ldots \rightarrow i_r=j$, then a monomial can be connected to this path as:

$$\alpha_P = \left(\prod_{i_k>j} x_{i_k}\right)\left(\prod_{i_l<i} y_{i_l}\right) \quad (4)$$

Theorem 1 ([4]). Suppose that G is a graph with vertices of n. An admissible path in G is then the set of binomials $GB(K_G)=\cup_{i<j}\{\alpha_P f_{ij} | P\}$, which is reduced Gröbner basis of K_G.

Theorem 2 ([4]). Let G be a graph with n vertices. Then the generating set of the initial ideal of the binomial edge ideal K_G is given by

$G(in_<(K_G)) = \bigcup_{i<j} \{\alpha_P x_i y_j \mid P \text{ is an admissible path in } G\}$.

4. Binomial Edge Ideal of Comb Graph CO_m and Its Properties

We consider a comb graph $G=CO_m$ with $m=2n$ vertices, where the vertex set is partitioned into two subsets: $1, 2, \ldots, \frac{m}{2}$ forming a path (the spine of the comb), and $\frac{m}{2}+1,\ldots,m$ representing pendant vertices (the teeth), each connected to exactly one corresponding vertex on the spine. In this labeling, vertex i is adjacent to vertex i+1 for $1\leq i<\frac{m}{2}$, and for each $1\leq i\leq\frac{m}{2}$, the vertex i is connected to the vertex $\frac{m}{2}+i$. The figure below illustrates the structure of this comb graph, where dotted edges represent continuation in the pattern.

In the following theorem, reduced Gröbner basis of K_G when G is comb graph CO_m. For this we will fix labeling as given in Figure 1.

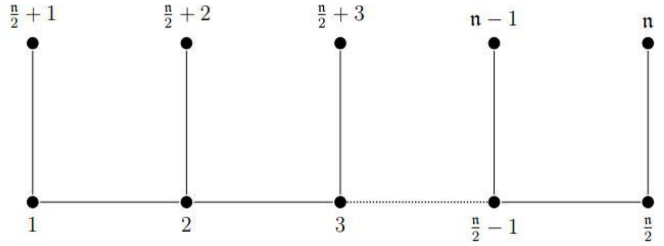

Figure 1. Comb graph $CO_m$with spine and pendant vertices.

Theorem 3. Let G be a comb graph CO_m with n vertices. Then reduced Gröbnerbasis of K_G is

$$GB(K_G)=\bigcup_{i=1}^{3} B_i \quad (5)$$

Where

$B_1=\{f_{ij}|\{i, j\}\in E(G)\}$,

$B_2=\{ \prod_{k=i-1}^{i-\theta} y_k\, f_{i,\frac{n}{2}+i-\theta}|2\leq i\leq\frac{n}{2},1\leq\theta\leq i-1\}$,

and $B_3=\{ \prod_{k=\theta}^{\theta+1} y_k\, f_{\frac{n}{2}+\theta,\frac{n}{2}+\theta+l}|1\leq\theta\leq\frac{n}{2}-1,\ 1\leq l\leq\frac{n}{2}-\theta\}$.

Proof. To prove this theorem we find the all possible admissible paths in comb graph CO_m and then associated monomials with these paths. If G is a comb graph with n vertices. Then we have the following cases:

Case 1. ($f_{ij}|\{i, j\}\in E(G)$).

It is obvious to see that $\alpha_P=1$.

Hence, $B_1=K_G$.

Case 2. ($2\leq i\leq\frac{n}{2}$and $j=\frac{n}{2}+i-\theta$, where $1\leq\theta\leq i-1$).

Clearly that P_θ: $i\rightarrow i-1\rightarrow\ldots\rightarrow i-\theta\rightarrow i-\theta+\frac{n}{2}$,

where $1\leq\theta\leq i-1$.

Now $\alpha_{P_\theta}=\prod_{k=i-\theta}^{i-1} y_k$.

Hence, $B_2=\{\alpha_{P_\theta} f_{i\frac{n}{2}+i-\theta}|2\leq i\leq\frac{n}{2},\ 1\leq\theta\leq i-1\}$.

Case 3. ($i=\frac{n}{2}+\theta$, $j=\frac{n}{2}+\theta+l$, $1\leq\theta\leq \frac{n}{2}-1$ and $1\leq l\leq\frac{n}{2}-\theta$).

Clearly, $P_{\theta,l}$: $\frac{n}{2}+\theta\rightarrow\theta\rightarrow\theta+1\ldots\rightarrow\theta+l\rightarrow\frac{n}{2}+l+\theta$,

Where $1\leq\theta\leq \frac{n}{2}-1$ and $1\leq l\leq\frac{n}{2}-\theta$.

Now $\alpha_{P_{\theta,1}}=\prod_{k=\theta}^{\theta+1} y_k$.

Hence, $B_3=\{\alpha_{P_{\theta,l}} f_{\frac{n}{2}+\theta,\frac{n}{2}+\theta+l}|1\leq\theta\leq \frac{n}{2}-1,\ 1\leq l\leq\frac{n}{2}-\theta\}$.

Now the theorem is concluded by using the above cases and Theorem 1.

Example 4. Let G be a comb graph CO_6 shown in Figure 2.

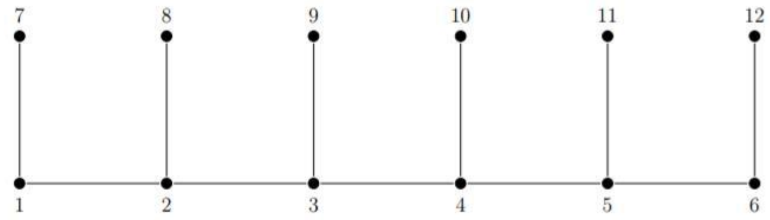

Figure 2. Comb graph CO_6 with 12 vertices, showing the spine and pendant vertices.

The comb graph CO_6 is formed by attaching a pendant vertex to each vertex of a path graph P_6. This results in a graph with 12 vertices and 11 edges. Its structure resembles a spine (the path) with evenly spaced teeth (pendants), hence the name "comb graph.

Then we have following cases for $GB(K_{CO_6})$.

(1) ($2\leq i\leq 6$, $j=6+i-\theta$ and $1\leq\theta\leq i-1$).

$\alpha_{P_\theta}=\prod_{k=i-\theta}^{i-1} y_k$, $1\leq\theta\leq i-1$. Here, each y_k is a variable from the polynomial ring $S=K[x_1, \ldots, x_{12}, y_1, \ldots, y_{12}]$ corresponding to the vertex k in the graph G.

$G_1=\{y_1f_{2,7}, y_2f_{3,8}, y_2y_1f_{3,7}, y_3f_{4,9}, y_3y_2f_{4,8}, y_3y_2y_1f_{4,7}, y_4f_{5,10}, y_4y_3f_{5,9}, y_4y_3y_2f_{5,8},$

$y_4y_3y_2y_1f_{5,7}, y_5f_{6,11}, y_5y_4f_{6,10}, y_5y_4y_3f_{6,9}, y_5y_4y_3y_2f_{6,8}, y_5y_4y_3y_2y_1f_{6,7}\}$.

(2) ($i=6+\theta$, $j=6+l+\theta$, $1\leq\theta\leq 5$, $1\leq l\leq 6-\theta$).

$\alpha_{P_{\theta,1}}=\prod_{k=\theta}^{\theta+1} y_k$, $1\leq\theta\leq 5$ and $1\leq l\leq 6-\theta$.

$G_2=\{y_1y_2f_{2,8}, y_1y_2y_3f_{7,9}, y_4y_3y_2y_1f_{7,10}, y_5y_4y_3y_2y_1f_{7,11}, y_6y_5y_4y_3y_2y_1f_{7,12},$

$y_3y_2f_{8,9}, y_2y_3y_4f_{8,10}, y_5y_4y_3y_2f_{8,11}, y_6y_5y_4y_3y_2f_{8,12}, y_4y_3f_{9,10}, y_3y_4y_5f_{9,11}, y_3y_4y_5y_6f_{9,12},$

$y_5y_4f_{10,11}, y_5y_4y_6f_{10,12}, y_5y_6f_{11,12}\}$.

(3) $G_3=K_{co_6}$.

Theorem 4. Let G be a comb graph CO_m with n vertices.

Then, $G(in_<(K_G))=\bigcup_{i=1}^{3} G_i$,

Where,

$G_1=\{x_iy_j|\{i, j\}\in E(G)\}$,

$G_2=\{\prod_{k=i-1}^{i-\theta} y_k\, x_i y_{\frac{n}{2}+i-\theta}|2\leq i\leq\frac{n}{2}, 1\leq\theta\leq i-1\}$,

and $G_3=\{\prod_{k=\theta}^{\theta+1} y_k\, x_{\frac{n}{2}+\theta} y_{\frac{n}{2}+\theta+l}|1\leq\theta\leq\frac{n}{2}-1, 1\leq l\leq\frac{n}{2}-\theta\}$.

Proof. Proof is trivial by Theorem 2.

Example 5. Let G be a graph CO_6 given in Figure 2.

Then we have following cases for $G(in_<(K_{CO_6}))$:

(1) ($2\leq i\leq 6$, $j=6+i-\theta$, $1\leq\theta\leq i-1$).

$\alpha_{P_\theta}=\prod_{k=i-\theta}^{i-1} y_k$, $1\leq\theta\leq i-1$.

$G_1=\{ y_1x_2y_7, y_2x_3y_8, y_2y_1x_3y_7, y_3x_4y_9, y_3y_2x_4y_8, y_3y_2y_1x_4y_7, y_4x_5y_{10},$

$y_4y_3x_5y_9, y_4y_3y_2x_5y_8, y_4y_3y_2y_1x_5y_7, y_5x_6y_{11}, y_5y_4x_6y_{10}, y_5y_4y_3x_6y_9, y_5y_4y_3y_2x_6y_8,$

$y_5y_4y_3y_2y_1x_6y_7\}$.

(2) ($i=6+\theta$, $j=6+l+\theta$, $1\leq\theta\leq 5$, $1\leq l\leq 6-\theta$).

$\alpha_{P_{\theta,l}}=\prod_{k=\theta}^{\theta+1} y_k$, $1\leq\theta\leq 5$, $1\leq l\leq 6-\theta$.

$G_2=\{y_1y_2x_7y_8, y_1y_2y_3x_7y_9, y_4y_3y_2y_1x_7y_{10}, y_5y_4y_3y_2y_1x_7y_{11}, y_6y_5y_4y_3y_2y_1x_7y_{12},$

$y_3y_2x_8y_9, y_2y_3y_4x_8y_{10}, y_5y_4y_3y_2x_8y_{11}, y_6y_5y_4y_3y_2x_8y_{12}, y_4y_3x_9y_{10}, y_3y_4y_5x_9y_{11}, y_3y_4y_5y_6x_9y_{12},$

$y_5y_4x_{10}y_{11}, y_5y_4y_6x_{10}y_{12}, y_5y_6x_{11}y_{12}\}.$

(3) $G_3=I_{cO_6}$.

5. Binomial Edge Ideal of Cross-Ladder Graph CL_m and Its Properties

In the following theorem, we will compute reduced Gröbner basis of K_G when G is cross-ladder graph as labeled in Figure 3.

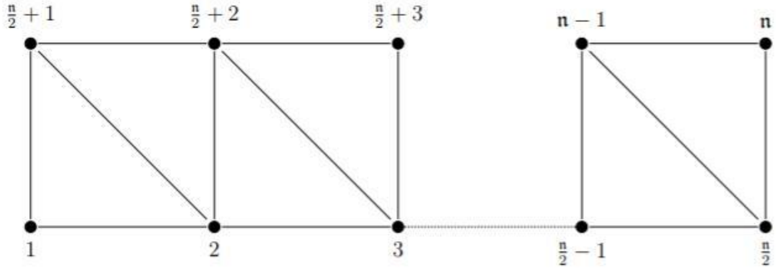

Figure 3. Cross-ladder graph CL_m with labeled vertices for Gröbner basis computation.

Theorem 5. Let G be cross-ladder graph CL_m with n vertices. Then reduced Gröbner basis of K_G is $GB(K_G)=\bigcup_{i=1}^{5} B_i$ where,

$B_1=\{f_{i\,j}|\{i, j\}\in E(G)\}$,

$B_2=\{\prod_{k=i-1}^{i-\theta} y_k\,(\prod_{k=i-\theta+\frac{n}{2}-1}^{i-\theta+\frac{n}{2}-(l-1)} x_k\,, l\geq 2)\, f_{i,\frac{n}{2}+i-\theta-l}|3\leq i\leq\frac{n}{2}, 1\leq\theta\leq i\text{-}2, 1\leq l\leq i-\theta\text{-}1\}$,

$B_3=\{\prod_{k=k-1}^{k-\theta} x_k\, f_{i,k-\theta-1}|3\leq i\leq\frac{n}{2}, 1\leq\theta\leq i\text{-}2, k=i+\frac{n}{2}\}$,

$B_4=\{\prod_{k=\theta+1}^{l} y_k\, f_{\frac{n}{2}+\theta,\frac{n}{2}+l}|1\leq\theta\leq\frac{n}{2}\text{-}2,\theta+2\leq l\leq\frac{n}{2}\}$,

and $B_5=\{\prod_{k=l}^{l+\theta} x_k\, f_{i,i+\theta+1}|1\leq i\leq\frac{n}{2}\text{-}2,1\leq\theta\leq\frac{n}{2}\text{-}i\text{-}1\}$.

Proof. To prove this theorem we need to find all the generalized admissible paths of the graph and their corresponding monomials. For this we have following cases:

Case 1. ($f_{ij}|\{i, j\}\in E(G)$).

It is obvious to see that $\alpha_P=1$.

Hence, $B_1=K_G$.

Case 2. ($3\leq i\leq\frac{n}{2}$, $j=\frac{n}{2}+i-\theta-l$, $1\leq\theta\leq i\text{-}2$ and $1\leq l\leq i-\theta\text{-}1$).

Clearly, $P_{\theta,\,l}$: $i\rightarrow i\text{-}1\rightarrow\ldots\rightarrow i-\theta\rightarrow i-\theta+\frac{n}{2}-l$,

Where $1\leq\theta\leq i\text{-}2$ and $1\leq l\leq i-\theta\text{-}1$.

Now $\alpha_{P_{\theta,\,l}}=\prod_{k=i-1}^{i-\theta} y_k \prod_{k=i-\theta+\frac{n}{2}-1}^{i-\theta+\frac{n}{2}-l+1} (x_k, l\geq 2)$.

Hence, $B_2=\{\alpha_{P_{\theta,l}} f_{i,\frac{n}{2}+i-\theta-l}|3\leq i\leq\frac{n}{2}, 1\leq\theta\leq i\text{-}2, 1\leq l\leq i\text{-}\theta\text{-}1\}$.

Case 3. ($3\leq i\leq\frac{n}{2}$, $j=k-\theta\text{-}1$ and $1\leq\theta\leq i\text{-}2$, where $k=i+\frac{n}{2}$).

It is obvious to see that

$P_{\theta,\,l}$: $i\rightarrow k\rightarrow k\text{-}1\ldots\rightarrow k\text{-}\theta\rightarrow k\text{-}\theta\text{-}l$,

Where $1\leq\theta\leq i\text{-}2$.

Now $\alpha_{P_{\theta,\,l}}=\prod_{m=k-1}^{k-\theta} x_m$.

Hence,

$B_3 = \{\alpha_{P_{\theta,l}} f_{i,k-\theta-1} | 3\le i\le \frac{n}{2}, 1\le\theta\le i-2\}$.

Case 4. ($i=\frac{n}{2}+\theta$, $=\frac{n}{2}+l$, $1\le\theta\le \frac{n}{2}-2$ and $\theta+2\le l\le\frac{n}{2}$).

It can be seen that

$P_{\theta,l}$: $\frac{n}{2}+\theta\rightarrow\theta+1\rightarrow\theta+2\ldots\rightarrow l\rightarrow l+\frac{n}{2}$.

Where $1\le\theta\le \frac{n}{2}-2$, $\theta+2\le l\le\frac{n}{2}$.

Now $\alpha_{P_{\theta,l}}=\prod_{k=\theta+1}^{l} y_k$.

Hence, $B_4=\{\alpha_{P_{\theta,l}} f_{\frac{n}{2}+\theta,\frac{n}{2}+l} | 1\le\theta\le\frac{n}{2}-2, \theta+2\le l\le\frac{n}{2}\}$.

Case 5. ($1\le i\le\frac{n}{2}-2$, $j=i+\theta+1$, $1\le\theta\le \frac{n}{2}-i-1$ and $l=i+\frac{n}{2}$).

It can be seen that

$P_{\theta,l}$: $i\rightarrow l\rightarrow l+1\ldots\rightarrow l+\theta\rightarrow i+\theta+1$,

Where $1\le\theta\le \frac{n}{2}-i-1$ and $l=i+\frac{n}{2}$.

Now $\alpha_{P_{\theta,l}}=\prod_{k=l}^{l+\theta} x_k$.

Hence, $B_5=\{\alpha_{P_{\theta,l}} f_{i,i+\theta+1} | 1\le i\le\frac{n}{2}-2, 1\le\theta\le\frac{n}{2}-i-1, l=i+\frac{n}{2}\}$.

Now the theorem is concluded by using the above cases and Theorem 1.

Example 6. Let G be a cross-ladder graph CL_5 as shown in Figure 4.

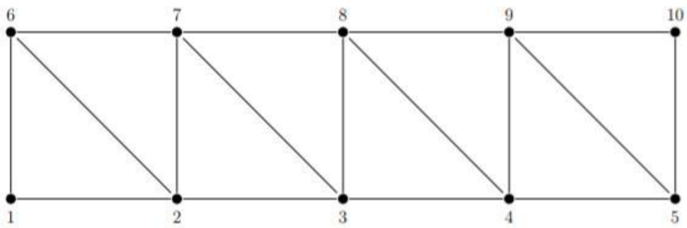

Figure 4. Cross-ladder graph CL_5 with labeled vertices.

Then $K_{CL_5}=\{x_1y_2-x_2y_1, x_1y_6-x_6y_1, x_2y_3-x_3y_2, x_2y_6-x_6y_2, x_2y_7-x_7y_2, x_3y_4-x_4y_3, x_3y_7-x_7y_3, x_3y_8-x_8y_3, x_4y_5-x_5y_4, x_4y_8-x_8y_4,$
$x_4y_9-x_9y_4, x_5y_9-x_9y_5, x_5y_{10}-x_{10}y_5, x_6y_7-x_7y_6, x_7y_8-x_8y_7, x_8y_9-x_9y_8, x_9y_{10}-x_{10}y_9\}$.

And

$GB(K_{CL_5})=\{y_4f_{5,8}, y_4x_8f_{5,7}, y_4x_8x_7f_{5,6}, y_4y_3f_{5,7}, y_4y_3y_2f_{5,6}, y_3f_{4,7}, y_3x_7f_{4,6}, y_3y_2f_{4,6}, y_2f_{3,6}, x_9f_{5,8}, x_9x_8f_{5,7}, x_9x_8x_7f_{5,6},$
$x_8f_{4,7}, x_8x_7f_{4,6}, x_7f_{3,6}, y_2y_3f_{6,8}, y_2y_3y_4f_{6,9}, y_2y_3y_4y_5f_{6,10}, y_3y_4f_{7,9}, y_3y_4y_5f_{7,10}, y_4y_5f_{8,10}, x_6x_7f_{1,3}, x_6x_7x_8f_{1,4},$
$x_6x_7x_8x_9f_{1,5}, x_7x_8f_{2,4}, x_7x_8x_9f_{2,5}, x_8x_9f_{3,5}\}$.

Theorem 6. Let G be cross-ladder graph CL_m with n vertices. Then $G(in_<(K_G)=\bigcup_{i=1}^{5} G_i$

Where,

$G_1=\{x_iy_j | \{i,j\}\in E(G)\}$,

$G_2=\{\prod_{k=i-1}^{i-\theta} y_k \, (\prod_{k=i-\theta+\frac{n}{2}-1}^{i-\theta+\frac{n}{2}-(l-1)} x_k, l\ge2) x_i y_{\frac{n}{2}+i-\theta-l} | 3\le i\le\frac{n}{2}, 1\le\theta\le i-2, 1\le l\le i-\theta-1\}$,

$G_3=\{\prod_{k=k-1}^{k-\theta} x_k \, x_i y_{k-\theta-1} | 3\le i\le\frac{n}{2}, 1\le\theta\le i-2, k=i+\frac{n}{2}\}$,

$G_4=\{\prod_{k=\theta+1}^{l} y_k \, x_i y_{\frac{n}{2}+l} | i=\frac{n}{2}+\theta, 1\le\theta\le\frac{n}{2}-2, \theta+2\le l\le\frac{n}{2}\}$,

and $G_5=\{\prod_{k=l}^{l+\theta} x_k \, x_i y_{i+\theta+1} | 1\le i\le\frac{n}{2}-2, 1\le\theta\le\frac{n}{2}-i-1\}$.

Proof. Proof is obvious by Theorem 2.

The admissible paths used in Theorems 5 and 6 are carefully constructed walks within the cross-ladder graph that respect its combinatorial symmetry and connectivity. These paths—ranging from simple edge connections to structured zig-zag and horizontal traversals—serve as the foundation for generating the reduced Gröbner basis and its initial ideal. Each family of binomials reflects a distinct class of admissible path, capturing the underlying geometry of the graph in algebraic form and highlighting the deep interplay between combinatorics and computational algebra.

Example 7. Let G be a CL_5 graph shown in Figure 4.

Then,

$K_{CL_5}=\{x_1y_2-x_2y_1, x_1y_6-x_6y_1, x_2y_3-x_3y_2, x_2y_6-x_6y_2, x_2y_7-x_7y_2, x_3y_4-x_4y_3, x_3y_7-x_7y_3, x_3y_8-x_8y_3, x_4y_5-x_5y_4, x_4y_8-x_8y_4,$

$x_4y_9-x_9y_4, x_5y_9-x_9y_5, x_5y_{10}-x_{10}y_5, x_6y_7-x_7y_6, x_7y_8-x_8y_7, x_8y_9-x_9y_8, x_9y_{10}-x_{10}y_9\}.$

And,

$G(in_<(K_{CL_5})=\{y_4x_5y_8, y_4x_8x_5y_7, y_4x_8x_7x_5y_6, y_4y_3x_5y_7, y_4y_3x_7x_5y_6, y_4y_3y_2x_5y_6, y_3x_4y_7, y_3x_7x_4y_6, y_3y_2x_4y_6, y_2x_3y_6,$

$x_9x_5y_8, x_9x_8x_5y_7, x_9x_8x_7x_5y_6, x_8x_4y_7, x_8x_7x_4y_6, x_7x_3y_6, y_2y_3x_6y_8, y_2y_3y_4x_6y_9, y_2y_3y_4y_5x_6y_{10}, y_3y_4x_7y_9,$

$y_3y_4y_5x_7y_{10}, y_4y_5x_8y_{10}, x_6x_7x_1y_3, x_6x_7x_8x_1y_4, x_6x_7x_8x_9x_1y_5, x_7x_8f_2y_4, x_7x_8x_9x_2y_5, x_8x_9x_3y_5\}.$

6. Binomial Edge Ideal of Sunlet Graph S_m and Its Properties

In the following theorem, reduced Gröbner basis of K_G when G is sunlet graph is given. For this first we will fix the labeling of sunlet graph as shown in Figure 5.

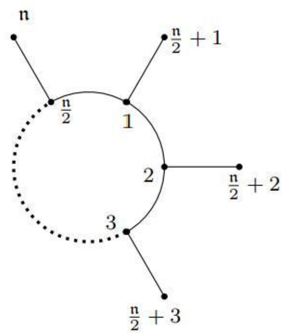

Figure 5. Sunlet graph with labeled vertices for Gröbner basis computation.

Theorem 7. Let G be n-sunlet graph S_m with n vertices. Then reduced Gröbner basis of K_G is: $GB(K_G)=\bigcup_{i=1}^{9} B_i(6)$,

Where,

$B_1=\{f_{ij}|\{i,j\}\in E(G)\}$,

$B_2=\{\prod_{k=\frac{n}{2}-\theta+1}^{\frac{n}{2}} x_k f_{1,\frac{n}{2}-\theta}|1\leq\theta\leq\frac{n}{2}-3\}$,

$B_3=\{\prod_{k=i-1}^{i\theta} y_k f_{i,\frac{n}{2}+i-\theta}|2\leq i\leq\frac{n}{2}-1, 1\leq\theta\leq i-1\}$,

$B_4=\{\prod_{k=1}^{i-1} y_k \prod_{k=\frac{n}{2}-\theta+1}^{\frac{n}{2}} x_k f_{i,\frac{n}{2}-\theta}|2\leq i\leq\frac{n}{2}-2, 0\leq\theta\leq\frac{n}{2}-i-2)\}$,

$B_5=\{\prod_{k=\frac{n}{2}-\theta}^{\frac{n}{2}-1} y_k f_{\frac{n}{2},n-\theta}|1\leq\theta\leq\frac{n}{2}-2\}$,

$B_6=\{\prod_{k=1}^{\theta} y_k f_{\frac{n}{2},\frac{n}{2}+\theta}|1\leq\theta\leq\frac{n}{2}-2\}$,

$B_7=\{\prod_{k=1}^{\theta} y_k f_{\frac{n}{2}+1,\frac{n}{2}+\theta}|2\leq\theta\leq\frac{n}{2}-1\}$,

$B_8=\{\prod_{k=\theta}^{l} y_k f_{\frac{n}{2}+\theta,\frac{n}{2}+1}|2\leq\theta\leq\frac{n}{2}-1, \theta+1\leq l\leq\frac{n}{2}\}$,

and $B_9=\{\prod_{k=1}^{l} y_k \prod_{k=\frac{n}{2}-\theta}^{\frac{n}{2}} f_{\frac{n}{2}+l, n-\theta} |0\leq\theta\leq\frac{n}{2}-l-2, 1\leq l\leq\frac{n}{2}-2\}$.

Proof. To prove the result we have to describe all possible admissible paths of n-sunlet graph G and corresponding monomials α_P of all i, j∈V(G) with i<j. For this we have the following cases:

Case 1. (f_{ij}|{i, j}∈E(G)).

It is obvious to see that α_P=1.

Case 2. (i=1 and j=$\frac{n}{2}$-θ, where1≤θ≤$\frac{n}{2}$-3).

It is obvious to see that P_θ: $1\to\frac{n}{2}\to\frac{n}{2}-1\ldots\to\frac{n}{2}-\theta$.

where $1\leq\theta\leq\frac{n}{2}-3$.

Now, $\alpha_{P_\theta}=\prod_{k=\frac{n}{2}}^{\frac{n}{2}-\theta+1}x_k$ for $1\leq\theta\leq\frac{n}{2}-3$.

Case 3. ($2\leq i\leq\frac{n}{2}-1$ and $j=\frac{n}{2}+i-\theta$, where $1\leq\theta\leq i-1$).

It is obvious to see that

P_θ: $i\to i-1\to\ldots\to i-\theta\to\frac{n}{2}+i-\theta$,

where $1\leq\theta\leq i-1$.

Now, $\alpha_{P_\theta}=\prod_{k=i-1}^{i-\theta}y_k$ for $1\leq\theta\leq i=-1$.

Case 4. ($2\leq i\leq\frac{n}{2}-2$ and $j=\frac{n}{2}-\theta$, where $0\leq\theta\leq\frac{n}{2}-i-2$).

It can be seen that

P_θ: $i\to i-1\to\ldots\to 1\to\frac{n}{2}\to\frac{n}{2}-1\to\ldots\to\frac{n}{2}-\theta$,

where $0\leq\theta\leq\frac{n}{2}-i-2$.

Now, $\alpha_{P_\theta}=\prod_{k=1}^{i-1}y_k\prod_{k=\frac{n}{2}}^{\frac{n}{2}-\theta+1}x_k$ for $0\leq\theta\leq\frac{n}{2}-i-2$.

Case 5. ($i=\frac{n}{2}$ and $j=n-\theta$, where $1\leq\theta\leq\frac{n}{2}-2$).

Clearly, P_θ: $\frac{n}{2}\to\frac{n}{2}-1\to\ldots\to\frac{n}{2}-\theta\to n-\theta$.

Where $1\leq\theta\leq\frac{n}{2}-2$.

Now, $\alpha_{P_\theta}=\prod_{k=\frac{n}{2}-\theta}^{\frac{n}{2}}y_k$ for $1\leq\theta\leq\frac{n}{2}-2$.

Case 6. ($i=\frac{n}{2}$ and $j=\frac{n}{2}+\theta$, where $1\leq\theta\leq\frac{n}{2}-2$).

It is obvious to see that

P_θ: $\frac{n}{2}\to 1\to\ldots\to\theta\to\frac{n}{2}+\theta$.

Where $1\leq\theta\leq\frac{n}{2}-2$.

Now, $\alpha_{P_\theta}=\prod_{k=1}^{\theta}y_k$ for $1\leq\theta\leq\frac{n}{2}-2$.

Case 7. ($i=\frac{n}{2}+1$ and $j=\frac{n}{2}+\theta$, where $2\leq\theta\leq\frac{n}{2}-1$).

It is clearly that P_θ: $\frac{n}{2}+1\to 1\to\ldots\to\theta\to\frac{n}{2}+\theta$.

Where $2\leq\theta\leq\frac{n}{2}-1$.

Now, $\alpha_{P_\theta}=\prod_{k=1}^{\theta}y_k$ for $2\leq\theta\leq\frac{n}{2}-1$.

Case 8. ($i=\frac{n}{2}+\theta$, $j=\frac{n}{2}+l$, where $2\leq\theta\leq \frac{n}{2}-1$ and $\theta+1\leq l\leq\frac{n}{2}$).

It is obvious to see that:

$P_{\theta,1}$: $\frac{n}{2}+\theta\rightarrow\theta\rightarrow\ldots\rightarrow l\rightarrow\frac{n}{2}+l$.

Where $2\leq\theta\leq \frac{n}{2}-1$ and $\theta+1\leq l\leq\frac{n}{2}$.

Now, $\alpha_{P_\theta}=\prod_{k=\theta}^{l} y_k$ for $2\leq\theta\leq \frac{n}{2}-1$ and $\theta+1\leq l\leq\frac{n}{2}$.

Case 9. ($i=\frac{n}{2}+l$, $j=n-\theta$, where $1\leq l\leq \frac{n}{2}-2$ and $0\leq\theta\leq\frac{n}{2}-l-2$).

It is obvious to see that:

$P_{\theta,1}$: $\frac{n}{2}+l\rightarrow l\rightarrow l-1\rightarrow\ldots\rightarrow 1\rightarrow\frac{n}{2}\rightarrow\frac{n}{2}-1\rightarrow\ldots\rightarrow\frac{n}{2}-\theta\rightarrow n-\theta$.

Where $1\leq l\leq \frac{n}{2}-2$ and $0\leq\theta\leq\frac{n}{2}-l-2$.

Now, $\alpha_{P_{\theta,l}}=\prod_{k=1}^{l}\prod_{k=\frac{n}{2}-\theta}^{\frac{n}{2}} y_k$ for $1\leq l\leq\frac{n}{2}-2$ and $0\leq\theta\leq\frac{n}{2}-l-2$.

Now the theorem is concluded by using the above cases and Theorem 1.

Example 8. Consider the sunlet graph S_5 shown in Figure 6.

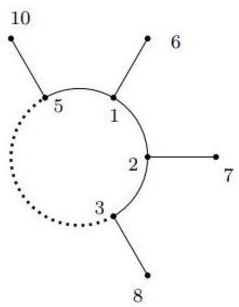

Figure 6. Sunlet graph S_5 with labeled vertices.

If

$K_{S_5}=\{x_1y_2-x_2y_1,x_1y_6-x_6y_1,x_1y_5-x_5y_1,x_2y_3-x_3y_2,x_2y_7-x_7y_2,x_3y_4-x_4y_3,x_3y_8-x_8y_3,x_4y_5-x_5y_4,x_4y_9-x_9y_4,x_5y_{10}-x_{10}y_5\}$.

Then we have following cases for $GB(K_{S_5})$:

(1) ($i=1$, $j=5-\theta$, where $1\leq\theta\leq2$).

$\alpha_{P_\theta}=\prod_{k=6-\theta}^{5} x_k$ for $1\leq\theta\leq2$.

Hence, $G_1=\{x_5f_{1,4},x_5x_4f_{1,3}\}$.

(2) ($2\leq i\leq4$, $j=5+i-\theta$ and $1\leq\theta\leq i-1$).

$\alpha_{P_\theta}=\prod_{k=i-\theta}^{i-1} y_k$ for $1\leq \theta \leq i-1$.

Hence, $G_2=\{y_2f_{3,7},y_1f_{2,6},y_2y_1f_{3,6},y_3f_{4,8},y_3y_2f_{4,7},y_3y_2y_1f_{4,6}\}$.

(3) ($2\leq i\leq3$, $j=5-\theta$ and $0\leq\theta\leq3-i$).

$\alpha_{P_\theta}=\prod_{k=1}^{i-1} y_k\prod_{k=6-\theta}^{5} x_k$ for $0\leq\theta\leq3-i$.

Hence, $G_3=\{f_{2,5}y_1,f_{2,4}y_1x_5,f_{3,5}y_1y_2\}$.

(4) ($i=5$, $j=10-\theta$, where $1\leq\theta\leq3$).

$\alpha_{P_\theta}=\prod_{k=5-\theta}^{5} y_k$ for $1\leq\theta\leq3$.

Hence, $G_4=\{y_4f_{5,9},y_4y_3f_{5,8},y_4y_3y_2f_{5,7}\}$.

(5) ($i=5$, $j=5+\theta$, where $1\leq\theta\leq 3$) .

$\alpha_{P_\theta}=\prod_{k=1}^{\theta} y_k$ for $1\leq\theta\leq 3$.

Hence, $G_5=\{y_1f_{5,6},y_1y_2f_{5,7},y_1y_3y_2f_{5,8}\}$.

(6) ($i=6$, $j=5+\theta$, where $2\leq\theta\leq 4$).

$\alpha_{P_\theta}=\prod_{k=1}^{\theta} y_k$ for $2\leq\theta\leq 4$.

Hence, $G_6=\{y_1y_2f_{6,7},y_1y_2y_3f_{6,8},y_1y_3y_2y_4f_{6,9}\}$.

(7) ($j=5+l$, $i=5+\theta$, where $2\leq\theta\leq 4$, $\theta+1\leq l\leq 5$).

$\alpha_{P_\theta}=\prod_{k=\theta}^{l} y_k$ for $2\leq\theta\leq 4$ and $\theta+1\leq l \leq 5$.

Hence, $G_7=\{y_2y_3f_{7,8},y_4y_2y_3f_{7,9},y_5y_3y_2y_4f_{7,10},y_3y_4f_{8,9},y_5y_3y_4f_{8,10},y_5y_4f_{9,10}\}$.

(8) ($i =5+l$, $j=10-\theta$, where $0\leq\theta\leq 3-l$, $1\leq l\leq 3$).

$\alpha_{P_\theta}=\prod_{k=1}^{l}\prod_{k=5-\theta}^{5} y_k$ for $0\leq\theta\leq 3-l$, $1\leq l\leq 3$.

Hence, $G_8=\{y_1y_5f_{6,10},y_1y_5y_4f_{6,9},y_1y_5y_4y_3f_{6,8},y_2y_1y_5f_{7,10},y_2y_1y_5y_4f_{7,9},y_1y_2y_3y_5f_{8,10}\}$.

(9)$G_9=K_{S_5}$.

Theorem 8. Let G be sunlet graph S_m with n vertices.

Then: $G(in_<(K_G)=\bigcup_{i=1}^{9} G_i(7)$,

Where

$G_1=\{x_iy_j|\{i,j\} \in E(G)\}$, $G_2=\{ \prod_{k=\frac{n}{2}-\theta+1}^{\frac{n}{2}} x_k\, x_1y_{\frac{n}{2}-\theta}|1\leq\theta\leq\frac{n}{2}-3\}$,

$G_3=\{ \prod_{k=i-1}^{i-\theta} y_k\, x_iy_{\frac{n}{2}+i-\theta}|2\leq i\leq\frac{n}{2}-1,1\leq\theta\leq i-1\}$,

$G_4=\{ \prod_{k=1}^{i-1} y_k \prod_{k=\frac{n}{2}-\theta+1}^{\frac{n}{2}} x_k\, x_iy_{\frac{n}{2}-\theta}|2\leq i\leq\frac{n}{2}-2,0\leq\theta\leq\frac{n}{2}-i-2\}$,

$G_5=\{ \prod_{k=\frac{n}{2}-\theta}^{\frac{n}{2}-1} y_k\, x_{\frac{n}{2}}y_{n-\theta}|1\leq\theta\leq\frac{n}{2}-2\}$,

$G_6=\{ \prod_{k=1}^{\theta} y_k\, x_{\frac{n}{2}}y_{\frac{n}{2}+\theta}|1\leq\theta\leq\frac{n}{2}-2\}$,

$G_7=\{ \prod_{k=1}^{\theta} y_k\, x_{\frac{n}{2}+1}y_{\frac{n}{2}+\theta}|2\leq\theta\leq\frac{n}{2}-1\}$,

$G_8=\{ \prod_{k=\theta}^{l} y_k\, x_{\frac{n}{2}+\theta}y_{\frac{n}{2}+l}|2\leq\theta\leq\frac{n}{2}-1,\theta+1\leq l\leq\frac{n}{2}\}$,

and $G_9=\{ \prod_{k=1}^{l} y_k \prod_{k=\frac{n}{2}-\theta}^{\frac{n}{2}} x_{\frac{n}{2}+1}\, y_{n-\theta}|0\leq\theta\leq\frac{n}{2}-l-2,1\leq l\leq\frac{n}{2}-2\}$.

Proof. Proof is trivial by Theorem 2.

Example 9. Consider the sunlet graph S_5 shown in Figure 6.

If

$K_{S_5}=\{x_1y_2-x_2y_1,x_1y_6-x_6y_1,x_1y_5-x_5y_1,x_2y_3-x_3y_2,x_2y_7-x_7y_2,x_3y_4-x_4y_3,x_3y_8-x_8y_3,x_4y_5-x_5y_4,x_4y_9-x_9y_4,x_5y_{10}-x_{10}y_5\}$.

Then we have following cases to for $G(in_<(K_{S_5}))$:

(1) ($i=1$, $j=5-\theta$, where$1\leq\theta\leq 2$).

$\alpha_{P_\theta}=\prod_{k=6-\theta}^{5} x_k$ for$1\leq\theta\leq 2$.

Hence, $G_1=\{x_5x_1y_4,x_5x_4x_1y_3\}$.

(2) ($2\leq i\leq 4$, $j=5+i-\theta$ and $1\leq\theta\leq i-1$).

$\alpha_{P_\theta}=\prod_{k=i-\theta}^{i-1} y_kfor1\leq\theta\leq i-1$.

Hence, $G_2=\{y_2x_3y_7,y_1x_2y_6,y_2y_1x_3y_6,y_3x_4y_8,y_3y_2x_4y_7,y_3y_2y_1x_4y_6\}$.

(3) ($2\leq\theta\leq 3$, where $1\leq\theta\leq 2$).

$\alpha_{P_\theta}=\prod_{k=1}^{i-1} y_k \prod_{k=6-\theta}^{5} x_k$ for $0\leq\theta\leq 3$-i.

Hence, $G_3=\{x_2y_5y_1, x_2y_4y_1x_5, x_3y_5y_1y_2\}$.

(4) (i=5, j=10-θ, where $1\leq\theta\leq 3$).

$\alpha_{P_\theta}=\prod_{k=5-\theta}^{5} y_k$ for $1\leq\theta\leq 3$.

Hence, $G_4=\{y_4x_5y_9, y_4y_3x_5y_8, y_4y_3y_2x_5y_7\}$.

(5) (i=5, j=5+θ, where $1\leq\theta\leq 3$).

$\alpha_{P_\theta}=\prod_{k=1}^{\theta} y_k$ for $1\leq\theta\leq 3$.

Hence, $G_5=\{y_1x_5y_6, y_1y_2x_5y_7, y_1y_3y_2x_5y_8\}$.

(6) (i =6, j=5+θ, where $2\leq\theta\leq 4$).

$\alpha_{P_\theta}=\prod_{k=1}^{\theta} y_k$ for $2\leq\theta\leq 4$.

Hence, $G_6=\{y_1y_2x_6y_7, y_1y_2y_3x_6y_8, y_1y_3y_2y_4x_6y_9\}$.

(7) (j=5+l, i=5+θ ,where $2\leq\theta\leq 4$, $\theta+1\leq l\leq 5$) .

$\alpha_{P_\theta}=\prod_{k=\theta}^{l} y_k$ for $2\leq\theta\leq 4$ and $\theta+1\leq l\leq 5$.

Hence, $G_7=\{y_2y_3x_7y_8, y_4y_2y_3x_7y_9, y_5y_3y_2y_4x_7y_{10}, y_3y_4x_8y_9, y_5y_3y_4x_8y_{10}, y_5y_4x_9y_{10}\}$.

(8) (i=5+l, j=10-θ, where $0\leq\theta\leq 3$-l, $1\leq l\leq 3$).

$\alpha_{P_\theta}=\prod_{k=1}^{l} \prod_{k=5-\theta}^{5} y_k$ for $0\leq\theta\leq 3$-l, $1\leq l\leq 3$.

Hence, $G_8=\{y_1y_5x_6y_{10}, y_1y_5y_4x_6y_9, y_1y_5y_4y_3x_6y_8, y_2y_1y_5x_7y_{10}, y_2y_1y_5y_4x_7y_9, y_1y_2y_3y_5x_8y_{10}\}$.

(9) $G_9=I_{S_5}$.

7. Discussion and Conclusion

In this work, we derived explicit reduced Gröbner bases and initial ideals of binomial edge ideals for three structured graph classes: comb graphs, cross-ladder graphs, and sunlet graphs. Our computations were performed under a fixed monomial order using the framework of admissible paths, yielding constructive formulas for each class.

The results highlight the growing complexity of Gröbner basis structures as the graphs transition from simple paths (in the comb graph) to intricate hybrid connections (in the cross-ladder and sunlet graphs). Particularly, the recurrence patterns in the associated monomials and binomials reflect the layered nature of these graphs.

Moreover, by systematically categorizing the admissible paths, we revealed how algebraic generators encode specific subgraph configurations. This connection offers future potential in characterizing the algebraic properties (like regularity or Betti numbers) directly from graph structure.

Future work can extend this investigation to disconnected graphs, graphs with loops, or weighted versions. It would also be valuable to explore applications of these results in conditional independence models in algebraic statistics, where binomial edge ideals frequently arise.

Conflicts of Interest

The authors declare no conflict of interest.

Generative AI Statement

The authors declare that no Gen AI was used in the creation of this manuscript.

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